

# SMOOTHNESS OF THE TRUNCATED DISPLAY FUNCTOR

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**ABSTRACT.** We show that to every  $p$ -divisible group over a  $p$ -adic ring one can associate a display by crystalline Dieudonné theory. For an appropriate notion of truncated displays, this induces a functor from truncated Barsotti-Tate groups to truncated displays, which is a smooth morphism of smooth algebraic stacks. As an application we obtain a new proof of the equivalence between infinitesimal  $p$ -divisible groups and nilpotent displays over  $p$ -adic rings, and a new proof of the equivalence due to Berthelot and Gabber between commutative finite flat group schemes of  $p$ -power order and Dieudonné modules over perfect rings.

## INTRODUCTION

The notion of displays over  $p$ -adic rings arises naturally both in Cartier theory and in crystalline Dieudonné theory.

In Cartier theory, displays are a categorised form of structure equations of Cartier modules of formal Lie groups. This is the original perspective in [Zi1]. Passing from a structure equation to the module corresponds to Zink's functor BT from displays to formal Lie groups, which induces an equivalence between nilpotent displays and  $p$ -divisible formal groups by [Zi1, La1]. The theory includes at its basis a description of the Dieudonné crystal of a  $p$ -divisible group  $\mathrm{BT}(\mathcal{P})$  in terms of the nilpotent display  $\mathcal{P}$ . We view this as a passage from Cartier theory to crystalline Dieudonné theory.

On the other hand, let  $G$  be a  $p$ -divisible group over a  $p$ -adic ring  $R$ , and let  $D$  be the covariant Dieudonné crystal of  $G$ . It is well-known that the Frobenius of  $D$  restricted to the Hodge filtration is divisible by  $p$ . If the ring of Witt vectors  $W(R)$  has no  $p$ -torsion, this gives a natural display structure on the value of  $D$  on  $W(R)$ . We show that this construction extends in a unique way to a functor from  $p$ -divisible groups to displays over an arbitrary  $p$ -adic ring  $R$

$$\Phi_R : (p\text{-div}/R) \rightarrow (\mathrm{disp}/R).$$

The proof uses that the stacks of truncated  $p$ -divisible groups are smooth algebraic stacks with smooth transition morphisms by [Il2], which implies that in a universal case the ring of Witt vectors has no  $p$ -torsion.

There is a natural notion of truncated displays over rings of characteristic  $p$ . While a display is given by an invertible matrix over  $W(R)$  if a suitable basis of the underlying module is fixed, a truncated display is given by an invertible matrix over the truncated Witt ring  $W_n(R)$  for a similar choice of basis. The functors  $\Phi_R$  induce functors from truncated Barsotti-Tate (BT) groups to truncated displays of the same level

$$\Phi_{n,R} : (p\text{-div}_n/R) \rightarrow (\mathrm{disp}_n/R).$$

For varying rings  $R$  of characteristic  $p$  they induce a morphism from the stack of truncated BT groups of level  $n$  to the stack of truncated displays of level  $n$ , which we denote by

$$\phi_n : \mathcal{BT}_n \rightarrow \mathcal{Disp}_n.$$

The following is the central result of this article.

**Theorem A.** *The morphism  $\phi_n$  is a smooth morphism of smooth algebraic stacks over  $\mathbb{F}_p$ , which is an equivalence on geometric points.*

Let us sketch the proof. The deformation theory of nilpotent displays together with the crystalline deformation theory of  $p$ -divisible groups implies that the restriction of the functor  $\Phi$  to infinitesimal  $p$ -divisible groups is formally étale in the sense that it induces an equivalence of infinitesimal deformations. It follows that the smooth locus of  $\phi_n$  contains all points of  $\mathcal{BT}_n$  that correspond to infinitesimal groups; since the smooth locus is open it must be all of  $\mathcal{BT}_n$ .

For a truncated BT group  $G$  we denote by  $\underline{\text{Aut}}^o(G)$  the sheaf of automorphisms of  $G$  which become trivial on the associated truncated display.

**Theorem B.** *Let  $G_1$  and  $G_2$  be truncated BT groups over a ring  $R$  of characteristic  $p$  with associated truncated displays  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . The group scheme  $\underline{\text{Aut}}^o(G_i)$  is commutative, infinitesimal, and finite flat over  $R$ . The natural morphism  $\underline{\text{Isom}}(G_1, G_2) \rightarrow \underline{\text{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$  is a torsor under  $\underline{\text{Aut}}^o(G_i)$  for each  $i$ .*

This is more or less a formal consequence of Theorem A. If  $G$  is a truncated BT group of dimension  $r$ , codimension  $s$ , and level  $n$ , one can show that the degree of  $\underline{\text{Aut}}^o(G)$  is equal to  $p^{rsn}$ . In particular, the functors  $\Phi_{n,R}$  are usually far from being an equivalence. The situation changes if one passes to the limit  $\Phi_R$ . Namely, we have the following application of Theorems A and B:

**Theorem C.** *For a  $p$ -adic ring  $R$ , the functor  $\Phi_R$  induces an equivalence between infinitesimal  $p$ -divisible groups and nilpotent displays over  $R$ .*

Let us sketch the argument. For a  $p$ -divisible group  $G$  over a ring  $R$  of characteristic  $p$  the situation is controlled by the projective limit of finite flat group schemes

$$\underline{\text{Aut}}^o(G) = \varprojlim_n \underline{\text{Aut}}^o(G[p^n]).$$

If the group  $G$  and its dual have a non-trivial étale part at some point of  $\text{Spec } R$ , one can see directly that  $\underline{\text{Aut}}^o(G)$  is non-trivial, which explains the restriction to infinitesimal groups in Theorem C. One has to show that  $\underline{\text{Aut}}^o(G)$  is trivial if  $G$  is infinitesimal. If  $\underline{\text{Aut}}^o(G)$  were non-trivial, the first homology of its cotangent complex would be non-trivial, which would contradict the fact that  $\Phi$  is formally étale for infinitesimal groups.

As a second application of Theorems A and B we obtain an alternative proof of the following result of Gabber.

**Theorem D.** *The category of  $p$ -divisible groups over a perfect ring  $R$  of characteristic  $p$  is equivalent to the category of Dieudonné modules over  $R$ .*

As in the case of perfect fields, a Dieudonné module over  $R$  is a projective  $W(R)$ -module  $M$  of finite type with a Frobenius-linear endomorphism  $F$  and a Frobenius<sup>-1</sup>-linear endomorphism  $V$  such that  $FV = p = VF$ . One deduces formally an equivalence between commutative finite flat group schemes of  $p$ -power order over  $R$  and an appropriate category of finite Dieudonné modules. Over perfect valuation rings this equivalence is proved by Berthelot [Be], and in general it is proved by Gabber by a reduction to the case of valuation rings. Theorem D follows from Theorems A and B since they show that the morphism  $\phi_n$  is represented by a morphism of groupoids of affine schemes which induces an isomorphism of the perfect hulls.

Finally, we study the relation between the functors  $\Phi_R$  and  $\text{BT}_R$ . One can form the composition

$$(p\text{-div}/R) \xrightarrow{\Phi_R} (\text{disp}/R) \xrightarrow{\text{BT}_R} (\text{formal groups}/R).$$

Here both functors induce inverse equivalences when restricted to formal  $p$ -divisible groups and nilpotent displays.

**Theorem E.** *For each  $p$ -divisible group  $G$  over a  $p$ -adic ring  $R$ , the formal group  $\text{BT}_R(\Phi_R(G))$  is naturally isomorphic to the formal completion  $\hat{G}$ .*

In other words, we have obtained a passage from crystalline Dieudonné theory to Cartier theory: The natural display structure on the Dieudonné crystal of  $G$ , viewed as a structure equation of a Cartier module, gives the Cartier module of  $\hat{G}$ .

The author thanks Th. Zink for many interesting discussions. Section 8.2 contains an erratum to [La1]. The author thanks O. Bültel for pointing out this mistake.

## CONTENTS

Introduction	1
1. Preliminaries	3
2. The display functor	6
3. Truncated displays	12
4. Smoothness of the truncated display functor	19
5. Classification of formal $p$ -divisible groups	23
6. Dieudonné theory over perfect rings	27
7. Small presentations	29
8. Relation with the functor BT	32
References	37

## 1. PRELIMINARIES

**1.1. Properties of ring homomorphisms.** All rings are commutative with a unit. Let  $f : A \rightarrow B$  be a ring homomorphism.

We call  $f$  *ind-étale* (resp. *ind-smooth*) if  $B$  can be written as a filtered direct limit of étale (resp. smooth)  $A$ -algebras. In the ind-étale case the transition maps in the filtered system are necessarily étale. We call  $f$  an

$\infty$ -smooth covering if there is a sequence of faithfully flat smooth ring homomorphisms  $A = B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots$  with  $B \cong \varinjlim B_i$ .

We call  $f$  *reduced* if  $f$  is flat and if the geometric fibres of  $f$  are reduced. This differs from EGA IV, 6.8.1, where in addition the fibres of  $f$  are assumed to be noetherian. If  $f$  is reduced, then for each reduced  $A$ -algebra  $A'$  the ring  $B \otimes_A A'$  is reduced. Every ind-smooth homomorphism is reduced.

Assume that  $A$  and  $B$  are noetherian. By the Popescu desingularisation theorem, [Po, Thm. 2.5] and [Sw],  $f$  is ind-smooth if and only if  $f$  is regular; recall that  $f$  is regular if  $f$  is flat and if  $B \otimes_A L$  is a regular ring when  $L$  is a finite extension of a residue field of a prime of  $A$ .

Without noetherian hypothesis again, we say that  $f$  is *quasi-étale* if the cotangent complex  $L_{B/A}$  is acyclic, and that  $f$  is *quasi-smooth* if the augmentation  $L_{B/A} \rightarrow \Omega_{B/A}$  is a quasi-isomorphism and if  $\Omega_{B/A}$  is a projective  $B$ -module. Quasi-smooth implies formally smooth, and quasi-étale implies formally étale; see [Ill, III, Proposition 3.1.1] and its proof.

**1.2. Affine algebraic stacks.** Let  $\text{Aff}$  be the category of affine schemes. Let  $\mathcal{X}$  be a category which is fibered in groupoids over  $\text{Aff}$ . For a topology  $\tau$  on  $\text{Aff}$ ,  $\mathcal{X}$  is called a  $\tau$ -stack if  $\tau$ -descent is effective for  $\mathcal{X}$ . We call  $\mathcal{X}$  an *affine algebraic stack* if  $\mathcal{X}$  is an fpqc stack, if the diagonal morphism  $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  is representable affine, and if there is an affine scheme  $X$  with a faithfully flat morphism  $X \rightarrow \mathcal{X}$ , called a presentation of  $\mathcal{X}$ . Equivalently,  $\mathcal{X}$  is the fpqc stack associated to a flat groupoid of affine schemes.

Let  $P$  be a property of ring homomorphisms which is stable under base change. A representable affine morphism of fpqc stacks is said to have the property  $P$  if its pull back to affine schemes has the property  $P$ . In particular, one can demand that an affine algebraic stack has a presentation with the property  $P$ , called a  $P$ -presentation.

Assume that the property  $P$  is stable under composition and satisfies the following descent condition: If a composition of ring homomorphisms  $v \circ u$  and  $v$  have the property  $P$  and if  $v$  is faithfully flat, then  $u$  has the property  $P$ . One example is  $P = \text{reduced}$ . Let  $\mathcal{X}$  be an affine algebraic stack which has a  $P$ -presentation  $X \rightarrow \mathcal{X}$ . A morphism of affine algebraic stacks  $\mathcal{X} \rightarrow \mathcal{Y}$  is said to have the property  $P$  if the composition  $X \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$  has the property  $P$ . This does not depend on the  $P$ -presentation of  $\mathcal{X}$ .

Let  $\mathcal{X}$  be an affine algebraic stack which has a reduced presentation  $X \rightarrow \mathcal{X}$ . We call  $\mathcal{X}$  *reduced* if  $X$  is reduced; this does not depend on the reduced presentation. In general, there is a maximal reduced closed substack  $\mathcal{X}_{\text{red}}$  of  $\mathcal{X}$ . Indeed, the inverse images of  $X_{\text{red}}$  under the two projections  $X \times_{\mathcal{X}} X \rightarrow X$  are equal because they coincide with  $(X \times_{\mathcal{X}} X)_{\text{red}}$ ; thus  $X_{\text{red}}$  descends to a substack of  $\mathcal{X}$ .

Assume that  $\mathcal{X}$  is a locally noetherian Artin algebraic stack and that  $Y$  is a locally noetherian scheme. We call a morphism  $Y \rightarrow \mathcal{X}$  *regular* if for a smooth presentation  $X \rightarrow \mathcal{X}$  the projection  $Y \times_{\mathcal{X}} X \rightarrow X$  is regular. This is independent of the smooth presentation of  $\mathcal{X}$ .

**1.3. The stack of  $p$ -divisible groups.** We fix a non-negative integer  $h$ . Let  $\mathcal{BT} = \mathcal{BT}^h$  be the stack of  $p$ -divisible groups of height  $h$ , viewed as a fibered category over the category of affine schemes. Thus for an affine

scheme  $X$ ,  $\mathcal{BT}(X)$  is the category with  $p$ -divisible groups of height  $h$  over  $X$  as objects and with isomorphisms of  $p$ -divisible groups as morphisms. Similarly, for each non-negative integer  $n$  let  $\mathcal{BT}_n = \mathcal{BT}_n^h$  be the stack of truncated Barsotti-Tate groups of height  $h$  and level  $n$ . This is an Artin algebraic stack of finite type over  $\mathbb{Z}$  with affine diagonal; see [W, Prop. 1.8] and [La1, Sec. 2]. The truncation morphisms

$$\tau_n : \mathcal{BT}_{n+1} \rightarrow \mathcal{BT}_n$$

are smooth and surjective by [Il2, Thm. 4.4 and Prop. 1.8]. Note that  $\mathcal{BT}_n$  has pure dimension zero since the dense open substack  $\mathcal{BT}_n \times \text{Spec } \mathbb{Q}$  is the classifying space of the finite group  $\text{GL}_h(\mathbb{Z}/p^n\mathbb{Z})$ .

**Lemma 1.6.** *The fibered category  $\mathcal{BT}$  is an affine algebraic stack in the sense of section 1.2. There is a presentation  $\pi : X \rightarrow \mathcal{BT}$  such that  $\pi$  and the compositions  $X \xrightarrow{\pi} \mathcal{BT} \xrightarrow{\tau} \mathcal{BT}_n$  for  $n \geq 0$  are  $\infty$ -smooth coverings; in particular  $X \rightarrow \text{Spec } \mathbb{Z}$  is an  $\infty$ -smooth covering.*

*Proof.* This follows from the properties of  $\mathcal{BT}_n$  and  $\tau_n$ , using that  $\mathcal{BT}$  is the projective limit of  $\mathcal{BT}_n$  for  $n \rightarrow \infty$ . More precisely, the diagonal of  $\mathcal{BT}$  is representable affine because a projective limit of affine schemes is affine. We choose smooth presentations  $\psi_n : Y_n \rightarrow \mathcal{BT}_n$  with affine  $Y_n$ , and define recursively another sequence of smooth presentations  $\pi_n : X_n \rightarrow \mathcal{BT}_n$  with affine  $X_n$  by  $X_1 = Y_1$  and  $X_{n+1} = Y_{n+1} \times_{\mathcal{BT}_n} X_n$ . Let

$$(1.1) \quad X = \varprojlim_n X_n = \varprojlim_n (X_n \times_{\mathcal{BT}_n} \mathcal{BT})$$

and let  $\pi : X \rightarrow \mathcal{BT}$  be the limit of the morphisms  $\pi_n$ . The transition maps in the second system in (1.1) are smooth and surjective because all  $\psi_n$  are smooth and surjective. Thus  $\pi$  is presentation of  $\mathcal{BT}$  and an  $\infty$ -smooth covering. The transition maps in the first system in (1.1) are smooth and surjective because the truncation morphisms  $\tau_n$  are smooth and surjective too. Thus  $X \rightarrow X_n \rightarrow \mathcal{BT}_n$  is an  $\infty$ -smooth covering.  $\square$

We refer to section 7 for presentations of  $\mathcal{BT} \times \text{Spec } \mathbb{Z}_p$  where the covering space is noetherian, and closer to  $\mathcal{BT}$  in some sense.

**1.4. Newton stratification.** In the following we write

$$\overline{\mathcal{BT}} = \mathcal{BT} \times \text{Spec } \mathbb{F}_p; \quad \overline{\mathcal{BT}_n} = \mathcal{BT}_n \times \text{Spec } \mathbb{F}_p.$$

We call a Newton polygon of height  $h$  a polygon that appears as the Newton polygon of a  $p$ -divisible group of height  $h$ .

**Lemma 1.7.** *For each Newton polygon  $\nu$  of height  $h$  there is a unique reduced closed substack  $\mathcal{BT}_\nu$  of  $\overline{\mathcal{BT}}$  such that the geometric points of  $\mathcal{BT}_\nu$  are the  $p$ -divisible groups with Newton polygon  $\preceq \nu$ .*

*Proof.* We consider a reduced presentation  $X \rightarrow \overline{\mathcal{BT}}$  with affine  $X$ , defined by the  $p$ -divisible group  $G$  over  $X$ . The points of  $X$  where  $G$  has Newton polygon  $\preceq \nu$  form a closed subset of  $X$ ; see [Ka, Thm. 2.3.1]. The corresponding reduced subscheme  $X_\nu$  of  $X$  descends to a reduced substack of  $\mathcal{BT}$  because the inverse images of  $X_\nu$  under the two projections  $X \times_{\mathcal{BT}} X \rightarrow X$  are reduced and coincide on geometric points, so they are equal.  $\square$

By a well-known boundedness principle, there is an integer  $N$  depending on  $h$  such that the Newton polygon of a  $p$ -divisible group  $G$  of height  $h$  is determined by its truncation  $G[p^N]$ .

**Lemma 1.8.** *For  $n \geq N$  there is a unique reduced closed substack  $\mathcal{BT}_{n,\nu}$  of  $\overline{\mathcal{BT}}_n$  such that we have a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{BT}_\nu & \longrightarrow & \mathcal{BT} \\ \downarrow & & \downarrow \tau \\ \mathcal{BT}_{n,\nu} & \longrightarrow & \mathcal{BT}_n \end{array}$$

where  $\tau$  is the truncation. In particular, the closed immersion  $\mathcal{BT}_\nu \rightarrow \mathcal{BT}$  is a morphism of finite presentation.

*Proof.* A reduced presentation  $X \rightarrow \overline{\mathcal{BT}}$  composed with  $\tau$  is a reduced presentation of  $\overline{\mathcal{BT}}_n$ . As in the proof of Lemma 1.7, the reduced subscheme  $X_\nu$  of  $X$  descends to a reduced substack of  $\overline{\mathcal{BT}}_n$ . Since  $\mathcal{BT}_n$  is of finite type, the immersion  $\mathcal{BT}_{n,\nu} \rightarrow \mathcal{BT}_n$  is a morphism of finite presentation.  $\square$

Along the same lines, one can consider the locus of infinitesimal groups:

**Lemma 1.9.** *There are unique reduced closed substacks  $\mathcal{BT}^\circ \subseteq \overline{\mathcal{BT}}$  and  $\mathcal{BT}_n^\circ \subseteq \overline{\mathcal{BT}}_n$  for  $n \geq 1$  such that the geometric points of  $\mathcal{BT}^\circ$  and  $\mathcal{BT}_n^\circ$  are precisely the infinitesimal groups. There is a Cartesian diagram*

$$\begin{array}{ccccc} \mathcal{BT}^\circ & \longrightarrow & \mathcal{BT}_{n+1}^\circ & \longrightarrow & \mathcal{BT}_n^\circ \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{BT} & \xrightarrow{\tau} & \mathcal{BT}_{n+1} & \xrightarrow{\tau} & \mathcal{BT}_n \end{array}$$

In particular, the closed immersion  $\mathcal{BT}^\circ \rightarrow \mathcal{BT}$  is of finite presentation.

*Proof.* Let  $G$  be a  $p$ -divisible group or truncated Barsotti-Tate group of positive level over an  $\mathbb{F}_p$ -scheme  $X$ . Since the points of  $X$  where the fibre of  $G$  is infinitesimal form a closed subset of  $X$ , the substacks  $\mathcal{BT}^\circ$  and  $\mathcal{BT}_n^\circ$  exist; see the proof of Lemma 1.7. The diagram is Cartesian since the truncation morphisms  $\tau$  are reduced, and since  $G$  is infinitesimal if and only if  $G[p]$  is infinitesimal. The vertical immersions are of finite presentation because  $\mathcal{BT}_n$  is of finite type.  $\square$

## 2. THE DISPLAY FUNCTOR

**2.1. Frame formalism.** We recall some constructions from [La2] and [La3]. Let  $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$  be a frame in the sense of [La2] with  $p\sigma_1 = \sigma$  on  $I$ .

In this article, the main example is the following. For a  $p$ -adic ring  $R$ , we denote by  $W(R)$  the ring of  $p$ -typical Witt vectors and by  $f$  and  $v$  the Frobenius and Verschiebung of  $W(R)$ . Let  $I_R = v(W(R))$  and let  $f_1 : I_R \rightarrow W(R)$  be the inverse of  $v$ . Then

$$\mathcal{W}_R = (W(R), I_R, R, f, f_1)$$

is a frame with  $pf_1 = f$ . Windows over  $\mathcal{W}_R$  in the sense of [La2] are (not necessarily nilpotent) displays over  $R$  in the sense of [Zi1] and [Me2].

For an  $S$ -module  $M$  let  $M^{(1)}$  be its  $\sigma$ -twist, and for a  $\sigma$ -linear map of  $S$ -modules  $\alpha : M \rightarrow N$  let  $\alpha^\sharp : M^{(1)} \rightarrow N$  be its linearisation. A filtered  $F$ - $V$ -module over  $\mathcal{F}$  is a quadruple  $(P, Q, F^\sharp, V^\sharp)$  where  $P$  is a projective  $S$ -module of finite type with a filtration  $IP \subseteq Q \subseteq P$  such that  $P/Q$  is projective over  $R$ , and where  $F^\sharp : P^{(1)} \rightarrow P$  and  $V^\sharp : P \rightarrow P^{(1)}$  are homomorphisms of  $S$ -modules with  $F^\sharp V^\sharp = p$  and  $V^\sharp F^\sharp = p$ . There is a functor

$$\Upsilon : (\mathcal{F}\text{-windows}) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathcal{F})$$

$$(P, Q, F, F_1) \mapsto (P, Q, F^\sharp, V^\sharp)$$

such that  $F^\sharp$  is the linearisation of  $F$ , and  $V^\sharp$  is determined by the relation  $V^\sharp(F_1(x)) = 1 \otimes x$  for  $x \in Q$ ; see [Zi1, Lemma 10] in the case of displays, and [La3, Lemma 2.3]. If  $S$  has no  $p$ -torsion,  $\Upsilon$  is fully faithful.

Assume that  $S$  and  $R$  are  $p$ -adic rings and that  $I$  is equipped with divided powers which are compatible with the canonical divided powers of  $p$ . For a  $p$ -divisible group  $G$  over  $R$  we denote by  $\mathbb{D}(G)$  the *covariant* Dieudonné crystal of  $G$ . By a standard construction, it gives rise to a functor

$$\Theta : (p\text{-div}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathcal{F})$$

$$G \mapsto (P, Q, F^\sharp, V^\sharp);$$

see [La3, Constr. 3.14]. Here  $P = \mathbb{D}(G)_{S \rightarrow R}$ , the submodule  $Q$  is the kernel of  $P \rightarrow \text{Lie}(G)$ , and the Frobenius and Verschiebung of  $G \otimes_R R/pR$  induce  $V^\sharp$  and  $F^\sharp$ . Note that  $F^\sharp$  is equivalent to a  $\sigma$ -linear map  $F : P \rightarrow P$ .

If  $S$  has no  $p$ -torsion, there is a unique  $\sigma$ -linear map  $F_1 : Q \rightarrow P$  such that  $(P, Q, F, F_1)$  is an  $\mathcal{F}$ -window that gives back  $(P, Q, F^\sharp, V^\sharp)$  when  $\Upsilon$  is applied; see [Ki, Lemma A.2] and [La3, Prop. 3.15]. In other words, there is a unique functor

$$\Phi : (p\text{-div}/R) \rightarrow (\mathcal{F}\text{-windows})$$

together with an isomorphism  $\Theta \cong \Upsilon \circ \Phi$ .

**2.2. The display functor.** Let  $R$  be a  $p$ -adic ring. The ideal  $I_R$  carries natural divided powers which are compatible with the canonical divided powers of  $p$ . Moreover the ring  $W(R)$  is  $p$ -adic; see [Zi1, Prop. 3]. Thus we have a functor  $\Theta$  for the frame  $\mathcal{W}_R$ , which we denote by

$$\Theta_R : (p\text{-div}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathcal{W}_R).$$

We also have a functor  $\Upsilon_R : (\text{disp}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathcal{W}_R)$ .

**Proposition 2.1.** *For each  $p$ -adic ring  $R$  there is a functor*

$$\Phi_R : (p\text{-div}/R) \rightarrow (\text{disp}/R)$$

*together with an isomorphism  $\Theta_R \cong \Upsilon_R \circ \Phi_R$  compatible with base change in  $R$ . This determines  $\Phi_R$  up to unique isomorphism.*

In other words, for each  $p$ -divisible group  $G$  over a  $p$ -adic ring  $R$  with  $\Theta_R(G) = (P, Q, F^\sharp, V^\sharp)$  there is a unique map  $F_1 : Q \rightarrow P$  which is functorial in  $G$  and  $R$  such that  $(P, Q, F, F_1)$  is a display which induces  $V^\sharp$ ; here  $F$  is defined by  $F(x) = F^\sharp(1 \otimes x)$ .

*Proof of Proposition 2.1.* Let  $X = \operatorname{Spec} A \xrightarrow{\pi} \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}_p$  be a reduced presentation, given by a  $p$ -divisible group  $G$  over  $A$ ; see Lemma 1.6. We write  $X \times_{\mathcal{BT} \times \operatorname{Spec} \mathbb{Z}_p} X = \operatorname{Spec} B$ . The rings  $A$  and  $B$  have no  $p$ -torsion; the rings  $A/pA$  and  $B/pB$  are reduced. Thus the  $p$ -adic completions  $\hat{A}$  and  $\hat{B}$  have no  $p$ -torsion and are reduced. In particular, the functors  $\Phi_{\hat{A}}$  and  $\Phi_{\hat{B}}$  exist and are unique, which implies that they commute with base change by arbitrary homomorphisms between  $\hat{A}$  and  $\hat{B}$ .

Since displays over a  $p$ -adic ring  $R$  are equivalent to compatible systems of displays over  $R/p^n R$  for  $n \geq 1$ , to prove the proposition it suffices to show that there is a unique functor  $\Phi_R$  if  $p$  is nilpotent in  $R$ . Let  $H$  be a  $p$ -divisible group over  $R$ . It defines a morphism  $\alpha : \operatorname{Spec} R \rightarrow \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}/p^m \mathbb{Z}$  for some  $m$ . We define  $S$  and  $T$  such that the following diagram has Cartesian squares, where  $\pi_1$  and  $\pi_2$  are the natural projections.

$$\begin{array}{ccccc} \operatorname{Spec} T & \xrightarrow{\psi_1} & \operatorname{Spec} S & \xrightarrow{\psi} & \operatorname{Spec} R \\ & \searrow \psi_2 & \downarrow \alpha' & & \downarrow \alpha \\ \operatorname{Spec} B & \xrightarrow[\pi_2]{\pi_1} & \operatorname{Spec} A & \xrightarrow{\pi} & \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}_p \end{array}$$

Then  $T \cong S \otimes_R S$  such that  $\psi_1$  and  $\psi_2$  are the projections, and  $\psi$  is faithfully flat. Let  $H_S = \psi^* H$ . We have descent data  $u : \pi_1^* G \cong \pi_2^* G$  and  $v : \psi_1^* H_S \cong \psi_2^* H_S$  and an isomorphism  $w : \alpha'^* G \cong H_S$  which preserves the descent data. Since  $p$  is nilpotent in  $R$ , the pair  $(\alpha', \alpha'')$  factors into

$$\begin{array}{ccccc} \operatorname{Spec} T & \xrightarrow{\hat{\alpha}''} & \operatorname{Spec} \hat{B} & \longrightarrow & \operatorname{Spec} B \\ \psi_2 \downarrow \downarrow \psi_1 & & \hat{\pi}_2 \downarrow \downarrow \hat{\pi}_1 & & \pi_2 \downarrow \downarrow \pi_1 \\ \operatorname{Spec} S & \xrightarrow{\hat{\alpha}'} & \operatorname{Spec} \hat{A} & \longrightarrow & \operatorname{Spec} A. \end{array}$$

The isomorphism  $u$  induces  $\hat{u} : \hat{\pi}_1^* G_{\hat{A}} \cong \hat{\pi}_2^* G_{\hat{A}}$ , and  $w$  induces an isomorphism  $\hat{w} : \hat{\alpha}'^* G_{\hat{A}} \cong H_S$  which transforms  $\hat{u}$  into  $v$ . The isomorphism  $\hat{w}$  induces an isomorphism of filtered  $F$ - $V$ -modules

$$\hat{\alpha}'^* \Theta_{\hat{A}}(G_{\hat{A}}) \cong \Theta_S(H_S).$$

Thus the operator  $F_1$  on  $\Theta_{\hat{A}}(G_{\hat{A}})$  given by  $\Phi_{\hat{A}}(G_{\hat{A}})$  induces an operator  $F_1$  on  $\Theta_S(H_S)$  which makes a display  $\Phi_S(H_S)$ . The descent datum  $\psi_1^* \Theta_S(H_S) \cong \psi_2^* \Theta_S(H_S)$  induced by  $v$  preserves  $F_1$  since  $\hat{w}$  transforms  $\hat{u}$  into  $v$  and since the isomorphism  $\pi_1^* \Theta_{\hat{A}}(G_{\hat{A}}) \cong \pi_2^* \Theta_{\hat{A}}(G_{\hat{A}})$  induced by  $\hat{u}$  preserves  $F_1$  by the uniqueness of  $\Phi_{\hat{B}}$ . By fpqc descent, cf. [Zi1, Thm. 37], the operator  $F_1$  on  $\Theta_S(H_S)$  descends to an operator  $F_1$  on  $\Theta_R(H)$  which makes a display  $\Phi_R(H)$ . This display is uniquely determined by the requirement that the functors  $\Phi_{\hat{B}}, \Phi_S, \Phi_R$  are compatible with base change by the given ring homomorphisms  $\hat{A} \rightarrow S \leftarrow R$ .

The construction implies that  $F_1$  is preserved under base change by homomorphisms  $R \rightarrow R'$  of  $p$ -adic rings and under isomorphisms of  $p$ -divisible groups over  $R$ . Since a homomorphism of  $p$ -divisible groups  $g : G \rightarrow G'$  can be encoded by the automorphism  $\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$  of  $G' \oplus G$ , it follows that  $F_1$  is also preserved under homomorphisms of  $p$ -divisible groups over  $R$ .  $\square$



**Proposition 2.2.** *A  $p$ -divisible group  $G$  over a  $p$ -adic ring  $R$  is infinitesimal (unipotent) if and only if the display  $\Phi_R(G)$  is nilpotent ( $F$ -nilpotent).*

*Proof.* The  $p$ -divisible group  $G$  is infinitesimal or unipotent if and only if the geometric fibres of  $G$  in points of characteristic  $p$  have this property; see [Me1, Chap. II, Prop. 4.4]. Similarly, a display  $\mathcal{P} = (P, Q, F, F_1)$  over  $R$  is nilpotent or  $F$ -nilpotent if and only if the geometric fibres of  $\mathcal{P}$  in points of characteristic  $p$  have this property. Indeed, let  $\bar{P}$  be the projective  $R/pR$ -module  $P/(I_R P + pP)$ . The display  $\mathcal{P}$  is nilpotent (resp.  $F$ -nilpotent) if and only if the homomorphism  $\bar{V}^\sharp : \bar{P} \rightarrow \bar{P}^{(1)}$  (resp.  $\bar{F}^\sharp : \bar{P}^{(1)} \rightarrow \bar{P}$ ) is nilpotent, which can be verified at the geometric points since  $\bar{P}$  is finitely generated. Thus the proposition follows from the case of perfect fields, which is well-known.  $\square$

*Remark 2.3.* There is a natural duality isomorphism  $\Phi_R(G^\vee) \cong \Phi_R(G)^t$ , where  $G^\vee$  is the Serre dual of  $G$ , and where  $^t$  denotes the dual display as in [Zi1, Def. 19]. Indeed, the crystalline duality theorem [BBM, 5.3] implies that the functor  $\Theta_R$  is compatible with duality, and the assertion follows from the uniqueness part of Proposition 2.1. See also [La3, Cor. 3.26].

**2.3. The extended display functor.** Assume that  $B \rightarrow R$  is a surjective homomorphism of  $p$ -adic rings whose kernel  $\mathfrak{b} \subset B$  is equipped with divided powers  $\delta$  that are compatible with the canonical divided powers of  $p$ . Then one can define a frame

$$\mathcal{W}_{B/R} = (W(B), I_{B/R}, R, f, \tilde{f}_1)$$

with  $p\tilde{f}_1 = f$ ; see [La3, Section 2.2]. Windows over  $\mathcal{W}_{B/R}$  are called displays for  $B/R$ . The ideal  $I_{B/R}$  carries natural divided powers, depending on  $\delta$ , which are compatible with the canonical divided powers of  $p$ ; see [La2, Section 2.7]. Thus there is a functor  $\Theta$  for  $\mathcal{W}_{B/R}$ , which we denote by

$$\Theta_{B/R} : (p\text{-div}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathcal{W}_{B/R}).$$

We also have  $\Upsilon_{B/R} : (\text{displays for } B/R) \rightarrow (\text{filtered } F\text{-}V\text{-mod. over } \mathcal{W}_{B/R})$ .

**Proposition 2.4.** *For each divided power extension  $B \rightarrow R$  of  $p$ -adic rings which is compatible with the canonical divided powers of  $p$ , there is a functor*

$$\Phi_{B/R} : (p\text{-div}/R) \rightarrow (\text{displays for } B/R)$$

*together with an isomorphism  $\Theta_{B/R} \cong \Upsilon_{B/R} \circ \Phi_{B/R}$  compatible with base change in  $(B, R, \delta)$ . This determines  $\Phi_{B/R}$  up to unique isomorphism.*

*Proof of Proposition 2.4.* We may assume that  $p$  is nilpotent in  $B$ . For a given  $p$ -divisible group  $H$  over  $R$  we choose a  $p$ -divisible group  $H_1$  over  $B$  which lifts  $H$ ; this is possible since  $B \rightarrow R$  is a nil-extension due to the divided powers. Necessarily we have to define  $\Phi_{B/R}(H)$  as the base change of  $\Phi_B(H_1)$  under the natural frame homomorphism  $\mathcal{W}_B \rightarrow \mathcal{W}_{B/R}$ . Here  $\Phi_B$  is well-defined by Proposition 2.1. We have to show that the operator  $F_1$  defined in this way on  $\Theta_{B/R}(H)$  does not depend on the choice of  $H_1$ ; then it follows easily that  $F_1$  is compatible with base change in  $B/R$  and commutes with homomorphisms of  $p$ -divisible groups over  $R$ .

As in the proof of Proposition 2.1, the assertion is reduced to a universal situation. Let  $X = \text{Spec } A \xrightarrow{\pi} \mathcal{B}\mathcal{T} \times \text{Spec } \mathbb{Z}_p$  be a presentation given by a

$p$ -divisible group  $G$  over  $X$  such that  $\pi$  and  $X \rightarrow \mathrm{Spec} \mathbb{Z}_p$  are  $\infty$ -smooth coverings; see Lemma 1.6. Let  $X' = \mathrm{Spec} A'$  be the  $p$ -adic completion of the divided power envelope of the diagonal  $X \rightarrow X \times_{\mathrm{Spec} \mathbb{Z}_p} X$  and let  $G_1, G_2$  be the inverse images of  $G$  under the two projections  $X' \rightarrow X$ . These are two lifts of  $G$  with respect to the diagonal morphism  $X \rightarrow X'$ . Since the divided power envelope of the diagonal of a smooth  $\mathbb{Z}_p$ -algebra has no  $p$ -torsion and since  $A$  is the direct limit of smooth  $\mathbb{Z}_p$ -algebras,  $A'$  has no  $p$ -torsion, and thus  $W(A')$  has no  $p$ -torsion. Thus the operators  $F_1$  on  $\Theta_{A'/A}(G)$  defined by  $\Phi_{A'}(G_1)$  and by  $\Phi_{A'}(G_2)$  are equal.

The given  $p$ -divisible group  $H$  over  $R$  defines a morphism  $\alpha : \mathrm{Spec} R \rightarrow \mathcal{BT} \times \mathrm{Spec} \mathbb{Z}_p$ . Since  $\pi$  is an  $\infty$ -smooth covering and since a surjective smooth morphism of schemes has a section étale locally in the base, we can find a ring  $R'$  and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec} R' & \xrightarrow{\alpha'} & X \\ \psi \downarrow & & \downarrow \pi \\ \mathrm{Spec} R & \xrightarrow{\alpha} & \mathcal{BT} \times \mathrm{Spec} \mathbb{Z}_p \end{array}$$

where  $\psi$  ind-étale and surjective. Since  $\mathrm{Spec} R \rightarrow \mathrm{Spec} B$  is a nil-immersion, there is a unique ind-étale and surjective morphism  $\mathrm{Spec} B' \rightarrow \mathrm{Spec} B$  which extends  $\psi$ . Since  $B'$  is flat over  $B$ , the given divided powers on the kernel of  $B \rightarrow R$  extend to divided powers on the kernel of  $B' \rightarrow R'$ . Let  $H_1$  and  $H_2$  be two lifts of  $H$  to  $B$  and let  $\beta_i : \mathrm{Spec} B' \rightarrow \mathcal{BT} \times \mathrm{Spec} \mathbb{Z}_p$  be the morphism given by  $H_i \otimes_B B'$  for  $i = 1, 2$ . We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec} R' & \xrightarrow{\alpha'} & X \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec} B' & \xrightarrow{\beta_i} & \mathcal{BT} \times \mathrm{Spec} \mathbb{Z}_p. \end{array}$$

Since  $\pi$  is an  $\infty$ -smooth covering and since a smooth morphism satisfies the lifting criterion of formal smoothness for arbitrary nil-immersions of affine schemes, there are morphisms  $\beta'_i : \mathrm{Spec} B' \rightarrow X$  for  $i = 1, 2$  such that in the preceding diagram both triangles commute. They define a morphism  $\beta' : \mathrm{Spec} B' \rightarrow X \times_{\mathrm{Spec} \mathbb{Z}_p} X$ , which factors uniquely over  $\beta'' : \mathrm{Spec} B' \rightarrow X'$ , and we have isomorphisms  $H_i \otimes_B B' \cong \beta''^* G_i$  that lift the given isomorphism  $H \otimes_R R' \cong \alpha'^* G$ . Thus the operators  $F_1$  on  $\Theta_{B'/R'}(H \otimes_R R')$  defined by  $\Theta_{B'}(H_1 \otimes_B B')$  and by  $\Theta_{B'}(H_2 \otimes_B B')$  are equal. Since  $W(B) \rightarrow W(B')$  is injective, it follows that the operators  $F_1$  on  $\Theta_{B/R}(H)$  defined by  $\Theta_B(H_1)$  and by  $\Theta_B(H_2)$  are equal as well.  $\square$

*Remark 2.5.* As in Remark 2.3 we have  $\Phi_{B/R}(G^\vee) \cong \Phi_{B/R}(G)^t$ .

**2.4. Consequences.** Let us recall the Dieudonné crystal of a nilpotent display. If  $B \rightarrow R$  is a divided power extension of  $p$ -adic rings which can be written as the projective limit of divided power extensions  $B_n \rightarrow R_n$  with  $p^n B_n = 0$ , for example if the divided powers are compatible with the canonical divided powers of  $p$ , the base change functor from nilpotent displays for  $B/R$  to nilpotent displays over  $R$  is an equivalence of categories by [Zi1,

Thm. 44]. Using this, one defines the Dieudonné crystal  $\mathbb{D}(\mathcal{P})$  of a nilpotent display  $\mathcal{P}$  over  $R$  as follows: If  $(\tilde{P}, \tilde{Q}, F, F_1)$  is the unique lift of  $\mathcal{P}$  to a display for  $B/R$ ,

$$\mathbb{D}(\mathcal{P})_{B/R} = \tilde{P}/I_B \tilde{P}.$$

By duality, the same applies to  $F$ -nilpotent displays.

*Remark 2.6.* It is well-known that the crystalline deformation theorem for  $p$ -divisible groups in [Me1] holds for not necessarily nilpotent divided powers if the groups are infinitesimal or unipotent; see [G1, Sec. 4] and [G2, p. 111].

More precisely, assume that  $B \rightarrow R = B/\mathfrak{b}$  is a divided power extension of rings in which  $p$  is nilpotent. We have a natural functor  $G \mapsto (H, V)$  from  $p$ -divisible groups  $G$  over  $B$  to  $p$ -divisible groups  $H$  over  $R$  together with a lift of the Hodge filtration of  $H$  to a direct summand  $V \subset \mathbb{D}(H)_{B/R}$ . Here  $\mathbb{D}(H)$  is the covariant Dieudonné crystal given by [BBM]. If the divided powers on  $\mathfrak{b}$  are nilpotent, this functor is an equivalence of categories by the deformation theorem [Me1, V.1.6] together with the comparison of the crystals of [Me1] and of [BBM] in [BM, 3.2.11]. For general divided powers, the functor induces an equivalence between unipotent (infinitesimal)  $p$ -divisible groups  $G$  over  $B$  and pairs  $(H, V)$  where  $H$  is unipotent (infinitesimal).

Let us indicate a proof of this fact. First, by the crystalline duality theorem [BBM, 5.3] it suffices to consider unipotent  $p$ -divisible groups.

For each commutative formal Lie group  $L = \mathrm{Spf} A$  over  $B$  there is a homomorphism  $\log_L : \mathrm{Ker}[L(B) \rightarrow L(R)] \rightarrow \mathrm{Ker}[\mathrm{Lie}(L) \rightarrow \mathrm{Lie}(L_R)]$  such that  $\log_{\hat{\mathbb{G}}_m}$  is given by the usual logarithm series. It can be described as follows: The Cartier dual  $L^* = \underline{\mathrm{Hom}}(L, \mathbb{G}_m)$  is an affine group scheme  $\mathrm{Spec} A^*$ . We have isomorphisms  $L \cong \underline{\mathrm{Hom}}(G^*, \mathbb{G}_m)$  and  $\mathrm{Lie}(L) \cong \underline{\mathrm{Hom}}(G^*, \mathbb{G}_a)$ , under which  $\log_L$  is induced by  $\log_{\hat{\mathbb{G}}_m}$ . If the divided powers on  $\mathfrak{b}$  are nilpotent,  $\log_G$  is an isomorphism; its inverse is given by the usual exponential series for  $\mathbb{G}_m$ . Let us call  $L$  unipotent if the augmentation ideal  $J \subset A^*$  is a nilideal. If  $L$  is unipotent, arbitrary divided powers on  $\mathfrak{b}$  induce nilpotent divided powers on  $\mathfrak{b}J$ , which implies that  $\log_L$  is an isomorphism again.

The construction of the Dieudonné crystal for nilpotent divided powers in [Me1] is based on the exponential of the formal completion of the universal vector extension  $EG$  of a  $p$ -divisible group  $G$  over  $B$ . If  $G$  is a unipotent  $p$ -divisible group, the formal completion of  $EG$  is a unipotent formal Lie group. Therefore, in the case of unipotent  $p$ -divisible groups, the construction of the Dieudonné crystal of [Me1] and the proof of the deformation theorem [Me1, V.1.6] are valid for not necessarily nilpotent divided powers. The comparison of crystals in [BM, 3.2.11] carries over to this case as well.  $\square$

Recall that we denote by  $\mathbb{D}(G)$  the covariant Dieudonné crystal of a  $p$ -divisible group  $G$ .

**Corollary 2.7.** *Let  $B \rightarrow R$  be a divided power extension of  $p$ -adic rings which is compatible with the canonical divided powers of  $p$ , and let  $G$  be a  $p$ -divisible group over  $R$  which is unipotent or infinitesimal. There is a natural isomorphism of projective  $B$ -modules*

$$\mathbb{D}(G)_{B/R} \cong \mathbb{D}(\Phi_R(G))_{B/R}.$$

*Proof.* If  $G$  is infinitesimal (unipotent) then  $\Phi_R(G)$  is nilpotent ( $F$ -nilpotent), and the display  $\Phi_{B/R}(G) = (\tilde{P}, \tilde{Q}, F, F_1)$  is the unique lift of  $\Phi_R(G)$  to a nilpotent ( $F$ -nilpotent) display for  $B/R$ . Since  $\tilde{P} = \mathbb{D}(G)_{W(B) \rightarrow R}$  and since the projection  $W(B) \rightarrow B$  is a homomorphism of divided power extensions of  $R$ , we get an isomorphism  $\mathbb{D}(G)_{B/R} \cong \tilde{P}/I_B \tilde{P} = \mathbb{D}(\Phi_R(G))_{B/R}$ .  $\square$

**Corollary 2.8.** *The restriction of  $\Phi_R$  to infinitesimal or unipotent  $p$ -divisible groups is formally étale, i.e. for a surjective homomorphism  $B \rightarrow R$  of rings in which  $p$  is nilpotent with nilpotent kernel, the category of infinitesimal (unipotent)  $p$ -divisible groups over  $B$  is equivalent to the category of such groups  $G$  over  $R$  together with a lift of  $\Phi_R(G)$  to a display over  $B$ .*

*Proof.* If the kernel of  $B \rightarrow R$  carries divided powers compatible with the canonical divided powers of  $p$ , in view of Corollary 2.7 the assertion follows from the crystalline deformation theorem of  $p$ -divisible groups (see Remark 2.6) and its counterpart for nilpotent (unipotent) displays [Zi1, Prop. 45]. Thus the corollary holds for  $B \rightarrow B/pB$  and for  $R \rightarrow R/pR$ . Hence we may assume that  $B$  is annihilated by  $p$ . Then  $B \rightarrow R$  is a finite composition of divided power extensions with trivial divided powers, which are compatible with the canonical divided powers of  $p$  since  $pB$  is zero.  $\square$

The following is a special case of Theorem 5.1.

**Corollary 2.9.** *Let  $R$  be a complete local ring with perfect residue field of characteristic  $p$ . The functor  $\Phi_R$  induces an equivalence between  $p$ -divisible groups over  $R$  with infinitesimal (unipotent) special fibre and displays over  $R$  with nilpotent ( $F$ -nilpotent) special fibre.*

*Proof.* Over perfect fields this is classical. The general case follows by Corollary 2.8 and by passing to the limit over  $R/\mathfrak{m}_R^n$ .  $\square$

### 3. TRUNCATED DISPLAYS

**3.1. Preliminaries.** For a  $p$ -adic ring  $R$  and a positive integer  $n$  let  $W_n(R)$  be the ring of truncated Witt vectors of length  $n$  and let  $I_{n,R} \subset W_n(R)$  be the kernel of the augmentation to  $R$ . The Frobenius of  $W(R)$  induces a ring homomorphism

$$f : W_{n+1}(R) \rightarrow W_n(R).$$

The inverse of the Verschiebung of  $W(R)$  induces a bijective  $f$ -linear map

$$f_1 : I_{n+1,R} \rightarrow W_n(R).$$

If  $R$  is an  $\mathbb{F}_p$ -algebra, the Frobenius of  $W(R)$  induces a ring endomorphism  $f$  of  $W_n(R)$ , and the ideal  $I_{n+1}(R)$  of  $W_{n+1}(R)$  is a  $W_n(R)$ -module.

**Definition 3.1.** A *pre-display* over an  $\mathbb{F}_p$ -algebra  $R$  is a sextuple

$$\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$$

where  $P$  and  $Q$  are  $W(R)$ -modules with homomorphisms

$$I_R \otimes_{W(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$$

and where  $F : P \rightarrow P$  and  $F_1 : Q \rightarrow P$  are  $f$ -linear maps such that the following relations hold: The compositions  $\iota\varepsilon$  and  $\varepsilon(1 \otimes \iota)$  are the multiplication homomorphisms, and we have  $F_1\varepsilon = f_1 \otimes F$ .

If  $P$  and  $Q$  are  $W_n(R)$ -modules, we call  $\mathcal{P}$  a pre-display of level  $n$ .

The axioms imply that  $F\iota = pF_1$ ; cf. [Zil, Eq. (2)].

Pre-displays over  $R$  form an abelian category  $(\text{pre-disp}/R)$  which contains the category of displays  $(\text{disp}/R)$  as a full subcategory. Pre-displays of level  $n$  over  $R$  form an abelian subcategory  $(\text{pre-disp}_n/R)$  of  $(\text{pre-disp}/R)$ . For a homomorphism of  $p$ -adic rings  $\alpha : R \rightarrow R'$ , the restriction of scalars defines a functor

$$\alpha^* : (\text{pre-disp}/R') \rightarrow (\text{pre-disp}/R).$$

It has a left adjoint  $\mathcal{P} \mapsto W(R') \otimes_{W(R)} \mathcal{P}$  given by the tensor product in each component. The restriction of  $\alpha^*$  to pre-displays of level  $n$  is a functor  $(\text{pre-disp}_n/R') \rightarrow (\text{pre-disp}_n/R)$  with left adjoint  $\mathcal{P} \mapsto W_n(R') \otimes_{W_n(R)} \mathcal{P}$ .

**3.2. Truncated displays.** Truncated displays of level  $n$  are pre-displays of level  $n$  with additional properties. We begin with the conditions imposed on the linear data of the pre-display. For an  $\mathbb{F}_p$ -algebra  $R$  let

$$J_{n+1,R} = \text{Ker}(W_{n+1}(R) \rightarrow W_n(R)).$$

**Definition 3.2.** A *truncated pair* of level  $n$  over an  $\mathbb{F}_p$ -algebra  $R$  is a quadruple  $\mathcal{B} = (P, Q, \iota, \varepsilon)$  where  $P$  and  $Q$  are  $W_n(R)$ -modules with homomorphisms

$$I_{n+1,R} \otimes_{W_n(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$$

such that the following properties hold.

- (1) The compositions  $\iota\varepsilon$  and  $\varepsilon(1 \otimes \iota)$  are the multiplication maps.
- (2) The  $W_n(R)$ -module  $P$  is projective of finite type.
- (3) The  $R$ -module  $\text{Coker}(\iota)$  is projective.
- (4) We have an exact sequence, where  $\bar{\varepsilon}$  is induced by  $\varepsilon$ :

$$(3.1) \quad 0 \rightarrow J_{n+1,R} \otimes_R \text{Coker}(\iota) \xrightarrow{\bar{\varepsilon}} Q \xrightarrow{\iota} P \rightarrow \text{Coker}(\iota) \rightarrow 0.$$

A *normal decomposition* for a truncated pair consists of projective  $W_n(R)$ -modules  $L \subseteq Q$  and  $T \subseteq P$  such that we have bijective homomorphisms

$$L \oplus T \xrightarrow{\iota+1} P, \quad L \oplus (I_{n+1,R} \otimes_{W_n(R)} T) \xrightarrow{1+\varepsilon} Q.$$

Each pair  $(L, T)$  of projective  $W_n(R)$ -modules of finite type defines a unique truncated pair for which  $(L, T)$  is a normal decomposition.

**Lemma 3.3.** *Every truncated pair  $\mathcal{B}$  admits a normal decomposition.*

*Proof.* Let  $\bar{L} = \text{Coker}(\varepsilon)$ , an  $R$ -module, and  $\bar{T} = \text{Coker}(\iota)$ , a projective  $R$ -module. The 4-term exact sequence (3.1) induces a short exact sequence

$$(3.2) \quad 0 \rightarrow \bar{L} \xrightarrow{\bar{\iota}} P/I_{n,R}P \rightarrow \bar{T} \rightarrow 0.$$

Thus  $\bar{L}$  is a projective  $R$ -module. Let  $L$  and  $T$  be projective  $W_n(R)$ -modules which lift  $\bar{L}$  and  $\bar{T}$ . Let  $L \rightarrow Q$  and  $T \rightarrow P$  be homomorphisms which commute with the obvious projections to  $\bar{L}$  and  $\bar{T}$ , respectively. The exact sequence (3.2) implies that the homomorphism  $\iota + 1 : L \oplus T \rightarrow P$  becomes

an isomorphism over  $R$ , so it is an isomorphism as both sides are projective. Let  $\mathcal{B}'$  be the truncated pair defined by  $(L, T)$ . We have a homomorphism of truncated pairs  $\mathcal{B}' \rightarrow \mathcal{B}$  such that the associated homomorphism of the 4-term sequences (3.1) is an isomorphism except possibly at  $Q$ . Hence it is an isomorphism by the 5-Lemma.  $\square$

**Definition 3.4.** A truncated display of level  $n$  over an  $\mathbb{F}_p$ -algebra  $R$  is pre-display  $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$  over  $R$  such  $(P, Q, \iota, \varepsilon)$  is a truncated pair of level  $n$  and such that the image of  $F_1$  generates  $P$  as a  $W_n(R)$ -module.

Let  $(\text{disp}_n/R)$  be the category of truncated displays of level  $n$  over  $R$ . This is a full subcategory of the abelian category  $(\text{pre-disp}_n/R)$ .

If  $(P, Q, \iota, \varepsilon)$  is a truncated pair with given normal decomposition  $(L, T)$ , the set of pairs  $(F, F_1)$  such that  $(P, Q, \iota, \varepsilon, F, F_1)$  is a truncated display is bijective to the set of  $f$ -linear isomorphisms  $\Psi : L \oplus T \rightarrow P$  such that  $\Psi|_L = F_1|_L$  and  $\Psi|_T = F|_T$ . If  $L$  and  $T$  are free  $W_n(R)$ -modules,  $\Psi$  is described by an invertible matrix over  $W_n(R)$ . This is analogous to the case of displays; see [Zi1, Lemma 9] and the subsequent discussion. The triple  $(L, T, \Psi)$  is called a normal representation of  $(P, Q, \iota, \varepsilon, F, F_1)$ .

Let  $k$  be a perfect field of characteristic  $p$ . A truncated Dieudonné module of level  $n$  over  $k$  is a triple  $(M, F, V)$  where  $M$  is a free  $W(k)$ -module of finite rank with an  $f$ -linear map  $F : M \rightarrow M$  and an  $f^{-1}$ -linear map  $V : M \rightarrow M$  such that  $FV = p = VF$ . If  $n = 1$  we require that  $\text{Ker } F = \text{Im } V$ , which is equivalent to  $\text{Ker } V = \text{Im } F$ .

**Lemma 3.5.** *Truncated displays of level  $n$  over a perfect field  $k$  are equivalent to truncated Dieudonné modules of level  $n$  over  $k$ .*

*Proof.* Multiplication by  $p$  gives an isomorphism  $W_n(k) \cong I_{n+1,k}$ . Thus truncated displays of level  $n$  are equivalent to quintuples  $\mathcal{P} = (P, Q, \iota, \varepsilon, F_1)$  where  $P$  and  $Q$  are free  $W_n(k)$ -modules with homomorphisms  $P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$  with  $\varepsilon\iota = p$  and  $\iota\varepsilon = p$  such that the sequence

$$Q \xrightarrow{\iota} P \xrightarrow{p^{-1}\varepsilon} Q \xrightarrow{\iota} P$$

is exact, and where  $F_1 : Q \rightarrow P$  is a bijective  $f$ -linear map. The exactness is automatic if  $n \geq 2$ . The operator  $F$  of the truncated display is given by  $F = F_1\varepsilon$ . Let  $V = \iota F_1^{-1}$ . The assignment  $\mathcal{P} \mapsto (P, F, V)$  is an equivalence between truncated displays and truncated Dieudonné modules.  $\square$

**Lemma 3.6.** *For a homomorphism of  $\mathbb{F}_p$ -algebras  $\alpha : R \rightarrow R'$  there is a unique base change functor*

$$\alpha_* : (\text{disp}_n/R) \rightarrow (\text{disp}_n/R')$$

*with a natural isomorphism*

$$\text{Hom}_{(\text{pre-disp}_n/R)}(\mathcal{P}, \alpha^* \mathcal{P}') \cong \text{Hom}_{(\text{disp}_n/R)}(\alpha_* \mathcal{P}, \mathcal{P}')$$

*for all truncated displays  $\mathcal{P}$  of level  $n$  over  $R$  and  $\mathcal{P}'$  of level  $n$  over  $R'$ .*

*Proof.* This is straightforward. In terms of normal representations,  $\alpha_*$  is given by  $(L, T, \Psi) \mapsto (W_n(R') \otimes_{W_n(R)} L, W_n(R') \otimes_{W_n(R)} T, f \otimes \Psi)$ .  $\square$

*Remark 3.7.* If  $\alpha$  is ind-étale, then  $W_n(R) \rightarrow W_n(R')$  is ind-étale, and the ideal  $v^m(I_{n-m,R'})$  is equal to  $W_n(R') \otimes_{W_n(R)} v^m(I_{n-m,R})$  for  $0 \leq m \leq n$ . This is proved in [LZ, Prop. A.8] if  $\alpha$  is étale, and the functor  $W_n$  preserves filtered direct limits of rings. As a consequence we obtain:

**Corollary 3.8.** *For each truncated display  $\mathcal{P}$  of level  $n$  over  $R$  there is a natural homomorphism of pre-displays over  $R'$*

$$W_n(R') \otimes_{W_n(R)} \mathcal{P} \rightarrow \alpha_* \mathcal{P}.$$

*If  $\alpha$  is ind-étale, this homomorphism is an isomorphism.*

*Proof.* In view of Remark 3.7, this follows from the proof of Lemma 3.6.  $\square$

**Lemma 3.9.** *Assume that  $\alpha : R \rightarrow R'$  is a faithfully flat ind-étale homomorphism of  $\mathbb{F}_p$ -algebras. If  $\mathcal{P}$  is a pre-display of level  $n$  over  $R$  such that  $\mathcal{P}' = W_n(R') \otimes_{W_n(R)} \mathcal{P}$  is a truncated display of level  $n$  over  $R'$ , then  $\mathcal{P}$  is a truncated display of level  $n$  over  $R$ .*

*Proof.* The pre-display  $\mathcal{P}$  is a truncated display if and only if  $P$  is projective of finite type over  $W_n(R)$ , the homomorphism  $F_1^\sharp : Q^{(1)} \rightarrow P$  is surjective, and the 4-term sequence (3.1) is exact. These properties descend from  $\mathcal{P}'$  to  $\mathcal{P}$  since in all components of  $\mathcal{P}$  and of (3.1), the passage from  $\mathcal{P}$  to  $\mathcal{P}'$  is given by the tensor product with the faithfully flat homomorphism  $W_n(\alpha)$ ; see Remark 3.7.  $\square$

**Lemma 3.10.** *For each  $\mathbb{F}_p$ -algebra  $R$  there are unique truncation functors*

$$\begin{aligned} \tau_n : (\text{disp}/R) &\rightarrow (\text{disp}_n/R), \\ \tau_n : (\text{disp}_{n+1}/R) &\rightarrow (\text{disp}_n/R), \end{aligned}$$

*together with a natural isomorphism*

$$\text{Hom}_{(\text{pre-disp}/R)}(\mathcal{P}, \mathcal{P}') \cong \text{Hom}_{(\text{disp}_n/R)}(\tau_n \mathcal{P}, \mathcal{P}')$$

*if  $\mathcal{P}$  is a display or truncated display of level  $n+1$  over  $R$  and if  $\mathcal{P}'$  is a truncated display of level  $n$  over  $R$ . The truncation functors are compatible with base change in  $R$ .*

*Proof.* Again this is straightforward. In terms of normal representations,  $\tau_n$  is given by  $(L, T, \Psi) \mapsto (W_n(R) \otimes_{W(R)} L, W_n(R) \otimes_{W(R)} T, f \otimes \Psi)$ .  $\square$

**Lemma 3.11.** *For an  $\mathbb{F}_p$ -algebra  $R$ , the category  $(\text{disp}/R)$  is the projective limit over  $n$  of the categories  $(\text{disp}_n/R)$ .*

*Proof.* It is easy to see that the truncation functor from displays over  $R$  to compatible systems of truncated displays of level  $n$  over  $R$  is fully faithful. For a given compatible system of truncated displays  $(\mathcal{P}_n)_{n \geq 1}$  we define  $\mathcal{P} = \varprojlim_n \mathcal{P}_n$  componentwise. The proof of Lemma 3.3 shows that each normal decomposition of  $\mathcal{P}_n$  can be lifted to a normal decomposition of  $\mathcal{P}_{n+1}$ . It follows easily that  $\mathcal{P}$  is a display.  $\square$

**3.3. Descent.** We recall the descent of projective modules over truncated Witt rings. Let  $R \rightarrow R'$  be a faithfully flat homomorphism of rings in which  $p$  is nilpotent and let  $R'' = R' \otimes_R R'$ . We denote by  $\mathcal{V}_n(R)$  the category of projective  $W_n(R)$ -modules of finite type and by  $\mathcal{V}_n(R'/R)$  the category of modules in  $\mathcal{V}_n(R')$  together with a descent datum relative to  $R \rightarrow R'$ .

**Lemma 3.12.** *The obvious functor  $\gamma : \mathcal{V}_n(R) \rightarrow \mathcal{V}_n(R'/R)$  is an equivalence.*

*Proof.* First we note that for each flat  $W_n(R)$ -module  $M$  the complex

$$C_n(M) = [0 \rightarrow M \rightarrow M \otimes_{W_n(R)} W_n(R') \rightarrow M \otimes_{W_n(R)} W_n(R'') \rightarrow \cdots]$$

is exact. Indeed, this is clear if  $n = 1$ , and in general  $C_n(M)$  is an extension of  $C_1(M \otimes_{W_n(R), f^{n-1}} R)$  and  $C_{n-1}(M \otimes_{W_n(R)} W_{n-1}(R))$ .

It follows that the functor  $\gamma$  is fully faithful. We have to show that  $\gamma$  is essentially surjective. If  $R'$  is a finite product of localisations of  $R$  then  $W_n(R) \rightarrow W_n(R')$  has the same property, and thus  $\gamma$  is an equivalence. Hence we may always pass to an open cover of  $\text{Spec } R$  by spectra of localisations of  $R$ . For  $M'$  in  $\mathcal{V}_n(R'/R)$  the descent datum induces a descent datum for the projective  $R'$ -module  $M'/I_{R'}M'$ , which is effective. By passing to a localisation of  $R$  we may assume that the descended  $R$ -module is free.

For fixed  $r$  let  $\mathcal{V}_n^o(R)$  be the category of modules  $M$  in  $\mathcal{V}_n(R)$  together with an isomorphism of  $R$ -modules  $\beta : R^r \cong M/I_R M$ ; homomorphisms in  $\mathcal{V}_n^o(R)$  preserve the  $\beta$ 's. In view of the preceding remarks it suffices to show that the obvious functor  $\mathcal{V}_n^o(R) \rightarrow \mathcal{V}_n^o(R'/R)$  is essentially surjective. Since every object of  $\mathcal{V}_n^o(R)$  is isomorphic to the standard object  $(W_n(R)^r, \text{id})$ , we have to show that all objects in  $\mathcal{V}_n^o(R'/R)$  are isomorphic. This means that for the sheaf of groups  $\underline{A}$  on the category of affine  $R$ -schemes defined by  $\underline{A}(\text{Spec } S) = \text{Aut}(W_n(S)^r, \text{id})$ , the Čech cohomology group  $\check{H}^1(R'/R, \underline{A})$  is trivial. This is true because  $\underline{A}$  has a finite filtration with quotients isomorphic to quasi-coherent modules.  $\square$

We turn to descent of truncated pairs. Let  $R \rightarrow R'$  be a faithfully flat homomorphism of  $\mathbb{F}_p$ -algebras. We denote by  $\mathcal{C}_n(R)$  the category of truncated pairs of level  $n$  over  $R$  and by  $\mathcal{C}_n(R'/R)$  the category of truncated pairs of level  $n$  over  $R'$  together with a descent datum relative to  $R \rightarrow R'$ .

**Lemma 3.13.** *The obvious functor  $\gamma : \mathcal{C}_n(R) \rightarrow \mathcal{C}_n(R'/R)$  is an equivalence.*

*Proof.* For a truncated pair  $\mathcal{B}$  over  $R$  we denote by  $\mathcal{B}', \mathcal{B}''$  the base change to  $R', R''$  etc. We have an exact sequence  $0 \rightarrow P \rightarrow P' \rightarrow P''$  by the proof of Lemma 3.12, and thus  $0 \rightarrow Q \rightarrow Q' \rightarrow Q''$  by the 4-term exact sequence (3.1). It follows easily that the functor  $\gamma$  is fully faithful. We show that  $\gamma$  is essentially surjective by a variant of the proof of Lemma 3.12. Again, the assertion holds if  $R'$  is a finite product of localisations of  $R$ , and thus we may pass to an open cover of  $\text{Spec } R$  defined by localisations. For  $\mathcal{B}'$  in  $\mathcal{C}_n(R'/R)$  the given descent datum induces a descent datum for the projective  $R$ -modules  $\text{Coker}(\iota)$  and  $\text{Coker}(\varepsilon)$ . By passing to a localisation of  $R$  we may assume that the descended  $R$ -modules are free.

For fixed  $r, s$  let  $\mathcal{C}_n^o(R)$  be the category of truncated pairs  $\mathcal{B}$  in  $\mathcal{C}_n(R)$  together with isomorphisms  $\beta_1 : R^r \cong \text{Coker}(\iota)$  and  $\beta_2 : R^s \cong \text{Coker}(\varepsilon)$  of



$R$ -modules; homomorphisms in  $\mathcal{C}_n^o(R)$  preserve the  $\beta_i$ . It suffices to show that  $\mathcal{C}_n^o(R) \rightarrow \mathcal{C}_n^o(R'/R)$  is essentially surjective. By Lemma 3.3 and its proof, all objects of  $\mathcal{C}_n^o(R)$  are isomorphic. For fixed  $(\mathcal{B}, \beta_1, \beta_2)$  in  $\mathcal{C}_n^o(R)$  with normal decomposition  $(L, T)$  the group  $\text{Aut}(\mathcal{B}, \beta_1, \beta_2)$  can be identified with the group of matrices  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  with  $A \in \text{Aut}(L)$ ,  $B \in \text{Hom}(T, L)$ ,  $C \in \text{Hom}(L, I_{n+1,R} \otimes_{W_n(R)} T)$ , and  $D \in \text{Aut}(T)$  such that  $A \equiv \text{id}$  and  $D \equiv \text{id}$  modulo  $I_R$ . The sheaf of groups  $\underline{A} = \underline{\text{Aut}}(\mathcal{B}, \beta_1, \beta_2)$  has a finite filtration with quotients isomorphic to quasi-coherent modules. Thus  $\check{H}^1(R'/R, \underline{A})$  is trivial, which implies that all objects of  $\mathcal{C}_n^o(R'/R)$  are isomorphic.  $\square$

**Proposition 3.14.** *Faithfully flat descent is effective for truncated displays over  $\mathbb{F}_p$ -algebras.*

*Proof.* By Lemmas 3.13 and 3.3 it suffices to show that for a given truncated pair  $\mathcal{B}$  over an  $\mathbb{F}_p$ -algebra  $R$  with a normal decomposition  $(L, T)$ , the truncated display structures on  $\mathcal{B}$  form an fpqc sheaf on the category of affine schemes over  $\text{Spec } R$ . This is true because these structures correspond to  $f$ -linear isomorphisms  $L \oplus T \rightarrow P$ .  $\square$

**3.4. Smoothness.** As in section 1.3 we fix a positive integer  $h$ . We denote by  $\mathcal{D}isp_n \rightarrow \text{Spec } \mathbb{F}_p$  the stack of truncated displays of level  $n$  and rank  $h$ . Thus  $\mathcal{D}isp_n(\text{Spec } R)$  is the groupoid of truncated displays of level  $n$  and rank  $h$  over  $R$ . The truncation functors induce morphisms

$$\tau_n : \mathcal{D}isp_{n+1} \rightarrow \mathcal{D}isp_n.$$

**Proposition 3.15.** *The fibered category  $\mathcal{D}isp_n$  is a smooth Artin algebraic stack of dimension zero over  $\mathbb{F}_p$  with affine diagonal. The morphism  $\tau_n$  is smooth and surjective of relative dimension zero.*

*Proof.* By Proposition 3.14,  $\mathcal{D}isp_n$  is an fpqc stack. In order to see that its diagonal is affine we have to show that for truncated displays  $\mathcal{P}_1$  and  $\mathcal{P}_2$  over an  $\mathbb{F}_p$ -algebra  $R$  the sheaf  $\underline{\text{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$  is represented by an affine scheme. By passing to an open cover of  $\text{Spec } R$  we may assume that  $\mathcal{P}_1$  and  $\mathcal{P}_2$  have normal decompositions by free modules. Then homomorphisms of the underlying truncated pairs are represented by an affine space. To commute with  $F$  and  $F_1$  is a closed condition, and a homomorphism of truncated pairs is an isomorphism if and only if it induces isomorphisms on  $\text{Coker}(\iota)$  and  $\text{Coker}(\varepsilon)$ , which means that two determinants are invertible. Thus  $\underline{\text{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$  is an affine scheme.

For each integer  $d$  with  $0 \leq d \leq h$  let  $\mathcal{D}isp_{n,d}$  be the substack of  $\mathcal{D}isp_n$  where the projective module  $\text{Coker}(\iota)$  has rank  $d$ . Let  $X_{n,d}$  be the functor on affine  $\mathbb{F}_p$ -schemes such that  $X_{n,d}(\text{Spec } R)$  is the set of invertible  $W_n(R)$ -matrices of rank  $h$ . Then  $X_{n,d}$  is an affine open subscheme of the affine space of dimension  $nh^2$  over  $\mathbb{F}_p$ . We define a morphism  $\pi_{n,d} : X_{n,d} \rightarrow \mathcal{D}isp_{n,d}$  such that the truncated display  $\pi_{n,d}(M)$  is given by the normal representation  $(L, T, \Psi)$ , where  $L = W_n(R)^{h-d}$  and  $T = W_n(R)^d$ , and  $M$  is the matrix representation of  $\Psi$ . Let  $G_{n,d}$  be the sheaf of groups such that  $G_{n,d}(R)$  is the group of invertible matrices  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  with  $A \in \text{Aut}(L)$ ,  $B \in \text{Hom}(T, L)$ ,  $C \in \text{Hom}(L, I_{n+1,R} \otimes_{W_n(R)} T)$ , and  $D \in \text{Aut}(T)$  for  $L$  and  $T$  as above. Then  $G_{n,d}$  is an affine open subscheme of the affine space of dimension  $nh^2$

over  $\mathbb{F}_p$ . The morphism  $\pi_{n,d}$  is a  $G_{n,d}$ -torsor. Thus  $\mathcal{D}isp_{n,d}$  and  $\mathcal{D}isp_n$  are smooth algebraic stacks of dimension zero over  $\mathbb{F}_p$ .

The truncation morphism  $\tau_n$  is smooth and surjective because it commutes with the obvious projection  $X_{n+1,d} \rightarrow X_{n,d}$ , which is smooth and surjective. The relative dimension of  $\tau$  is the difference of the dimensions of its source and target, which are both zero.  $\square$

Let  $\mathcal{D}isp \rightarrow \text{Spec } \mathbb{F}_p$  be the stack of displays over  $\mathbb{F}_p$ -algebras.

**Corollary 3.16.** *The fibered category  $\mathcal{D}isp$  is an affine algebraic stack over  $\mathbb{F}_p$ , which has a presentation  $\pi : X \rightarrow \mathcal{D}isp$  such that  $\pi$  and the compositions  $X \rightarrow \mathcal{D}isp \xrightarrow{\tau} \mathcal{D}isp_n$  for  $n \geq 0$  are  $\infty$ -smooth coverings.*

*Proof.* By Lemma 3.11,  $\mathcal{D}isp$  is the projective limit over  $n$  of  $\mathcal{D}isp_n$ . Thus the corollary follows from Proposition 3.15 by the proof of Lemma 1.6.  $\square$

**3.5. Nilpotent truncated displays.** Let  $R$  be an  $\mathbb{F}_p$ -algebra. For each truncated display  $\mathcal{P} = (P, Q, \iota, \varepsilon, F, F_1)$  of positive level  $n$  over  $R$  there is a unique homomorphism

$$V^\sharp : P \rightarrow P^{(1)} = W_n(R) \otimes_{f, W_n(R)} P$$

such that  $V^\sharp(F_1(x)) = 1 \otimes x$  for  $x \in Q$ . If  $F^\sharp : P^{(1)} \rightarrow P$  denotes the linearisation of  $F$ , we have  $F^\sharp V^\sharp = p$  and  $V^\sharp F^\sharp = p$ . This is analogous to the case of displays; see [Zil, Lemma 10]. The construction of  $V^\sharp$  is compatible with truncation. We call  $\mathcal{P}$  nilpotent if for some  $n$  the  $n$ -th iterate of  $V^\sharp$

$$P \rightarrow P^{(1)} \rightarrow \dots \rightarrow P^{(n)}$$

is zero. Since the ideal  $I_{n,R}$  is nilpotent,  $\mathcal{P}$  is nilpotent if and only if the truncation  $\tau_1(\mathcal{P})$  of level 1 is nilpotent. A display over  $R$  is nilpotent if and only if all its truncations are nilpotent.

**Lemma 3.17.** *There are unique reduced closed substacks  $\mathcal{D}isp^o \subseteq \mathcal{D}isp$  and  $\mathcal{D}isp_n^o \subseteq \mathcal{D}isp_n$  for  $n \geq 1$  such that the geometric points of  $\mathcal{D}isp^o$  and  $\mathcal{D}isp_n^o$  are precisely the nilpotent (truncated) displays. There is a Cartesian diagram*

$$\begin{array}{ccccc} \mathcal{D}isp^o & \longrightarrow & \mathcal{D}isp_{n+1}^o & \longrightarrow & \mathcal{D}isp_n^o \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}isp & \xrightarrow{\tau} & \mathcal{D}isp_{n+1} & \xrightarrow{\tau} & \mathcal{D}isp_n. \end{array}$$

*In particular, the closed immersion  $\mathcal{D}isp^o \rightarrow \mathcal{D}isp$  is of finite presentation.*

*Proof.* Over a field, a truncated display of level 1 and rank  $h$  is nilpotent if and only if the  $h$ -th iterate of  $V^\sharp$  vanishes. Thus for a display or truncated display of positive level  $\mathcal{P}$  over an  $\mathbb{F}_p$ -algebra  $R$  the points of  $\text{Spec } R$  where  $\mathcal{P}$  is nilpotent form a closed subset. Since  $\mathcal{D}isp$  and  $\mathcal{D}isp_n$  have reduced presentations and since the truncation morphisms  $\tau$  are reduced, the existence of the reduced closed substacks  $\mathcal{D}isp^o$  and  $\mathcal{D}isp_n^o$  and the Cartesian diagram follow; cf. Lemma 1.9.  $\square$

## 4. SMOOTHNESS OF THE TRUNCATED DISPLAY FUNCTOR

**4.1. The truncated display functor.** We begin with the observation that the display functors  $\Phi_R$  induce truncated display functors on each level. Recall that  $(p\text{-div}_n/R)$  is the category of truncated Barsotti-Tate groups of level  $n$  over  $R$ , and  $(\text{disp}_n/R)$  is the category of truncated displays of level  $n$  over  $R$ , which is defined if  $R$  is an  $\mathbb{F}_p$ -algebra.

**Proposition 4.1.** *For each  $\mathbb{F}_p$ -algebra  $R$  and each positive integer  $n$  there is a unique functor*

$$\Phi_{n,R} : (p\text{-div}_n/R) \rightarrow (\text{disp}_n/R)$$

*which is compatible with base change in  $R$  and with the truncation functors from  $n+1$  to  $n$  on both sides such that the system  $(\Phi_{n,R})_{n \geq 1}$  induces  $\Phi_R$  in the projective limit.*

*Proof.* Let  $(p\text{-div}_n/R)'$  be the category of all  $G$  in  $(p\text{-div}_n/R)$  which can be written as the kernel of an isogeny of  $p$ -divisible groups  $H_0 \rightarrow H_1$  over  $R$ . First we define a functor

$$\Phi'_{n,R} : (p\text{-div}_n/R)' \rightarrow (\text{pre-disp}_n/R).$$

For  $G$  in  $(p\text{-div}_n/R)'$  we choose an isogeny of  $p$ -divisible group  $H_0 \rightarrow H_1$  with kernel  $G$  and define

$$\Phi'_{n,R}(G) = \text{Coker}(\tau_n \Phi_R(H_0) \rightarrow \tau_n \Phi_R(H_1)),$$

where  $\Phi_R$  is given by Proposition 2.1, and where  $\tau_n$  is the truncation from displays to truncated displays of level  $n$ . If  $g : G \rightarrow G'$  is a homomorphism in  $(p\text{-div}_n/R)'$  such that  $G$  is the kernel of  $H_0 \rightarrow H_1$  and  $G'$  is the kernel of  $H'_0 \rightarrow H'_1$ , we define  $\Phi'_{n,R}(g)$  as follows. Let  $H''_0 = H_0 \times H'_0$ , let  $G \rightarrow H''_0$  be given by  $(1, g)$ , and let  $H''_1 = H''_0/G$ . The projections  $H_0 \leftarrow H''_0 \rightarrow H'_0$  extend uniquely to homomorphisms of complexes  $H_* \leftarrow H''_* \rightarrow H'_*$ , where the first arrow is a quasi-isomorphism. This means that its cone is exact, which is preserved by  $\tau_n \circ \Phi_R$ . Thus the homomorphisms of complexes

$$\tau_n \Phi_R(H_*) \leftarrow \tau_n \Phi_R(H''_*) \rightarrow \tau_n \Phi_R(H'_*)$$

induce a homomorphism of pre-displays  $\Phi'_{n,R}(g) : \Phi'_{n,R}(G) \rightarrow \Phi'_{n,R}(G')$  on the cokernels. It is easy to verify that  $\Phi'_{n,R}$  is a well-defined functor, which is independent of the chosen isogenies; see also [La2, 8.5] and [La3, 4.1].

Since  $\Phi_R$  and  $\tau_n$  are compatible with base change in  $R$ , for a ring homomorphism  $\alpha : R \rightarrow R'$  and for  $G$  in  $(p\text{-div}_n/R)'$  we get a natural homomorphism of pre-displays over  $R'$

$$u' : W_n(R') \otimes_{W_n(R)} \Phi'_{n,R}(G) \rightarrow \Phi'_{n,R'}(G \otimes_R R').$$

If  $\alpha$  is ind-étale, Corollary 3.8 implies that  $u'$  is an isomorphism. Since  $W_n$  preserves ind-étale coverings of rings by [LZ, Prop. A.8]; cf. Remark 3.7, ind-étale descent is effective for pre-displays of level  $n$ .

Assume that  $G$  is the  $p^n$ -torsion of a  $p$ -divisible group  $H$  over  $R$ . Then we can use the isogeny  $p^n : H \rightarrow H$  in the construction of  $\Phi'_{n,R}(G)$ . Since  $p^n$  annihilates  $W_n(R)$ , it follows that  $\Phi'_{n,R}(G) = \tau_n \Phi_R(H)$ . In particular, in this case the pre-display  $\Phi'_{n,R}(G)$  is a truncated display of level  $n$ .

For each  $G \in (p\text{-div}_n/R)$  there is a sequence of faithfully flat smooth ring homomorphisms  $R = R_0 \rightarrow R_1 \rightarrow R_2 \cdots$  such that, if we write  $R' = \varinjlim R_i$ , the group  $G \otimes_R R'$  is the  $p^n$ -torsion of a  $p$ -divisible group over  $R'$ ; see Lemma 1.6. Since a surjective smooth morphism of schemes has a section étale locally in the base, we can assume that  $R \rightarrow R'$  is ind-étale. By Lemma 3.9 it follows that the image of  $\Phi'_{n,R}$  lies in  $(\text{disp}_n/R)$ , and by ind-étale descent we get a unique extension of  $\Phi'_{n,R}$  to a functor  $\Phi_{n,R}$  as in the proposition which is compatible with ind-étale base change in  $R$ .

For an arbitrary homomorphism  $\alpha : R \rightarrow R'$  of  $\mathbb{F}_p$ -algebras and for  $G$  in  $(p\text{-div}_n/R)$ , by ind-étale descent, the above homomorphisms  $u'$  induce a homomorphism of pre-displays over  $R'$

$$u : W_n(R') \otimes_{W_n(R)} \Phi_{n,R}(G) \rightarrow \Phi_{n,R'}(G \otimes_R R').$$

Since  $\Phi_{n,R}(G)$  and  $\Phi_{n,R'}(G \otimes_R R')$  are truncated displays,  $u$  induces a base change homomorphism of truncated displays over  $R'$

$$\tilde{u} : \alpha_* \Phi_{n,R}(G) \rightarrow \Phi_{n,R'}(G \otimes_R R').$$

We claim that  $\tilde{u}$  is an isomorphism. If  $G$  is the  $p^n$ -torsion of a  $p$ -divisible group  $H$  over  $R$ , this is true because then  $\Phi'_{n,R}(G) = \tau_n \Phi_R(H)$ . The general case follows by passing to an ind-étale covering of  $R$ .  $\square$

*Remark 4.2.* For the construction of  $\Phi_{n,R}$  as a functor from  $(p\text{-div}_n/R)$  to  $(\text{pre-disp}_n/R)$  one can also use the theorem of Raynaud [BBM, 3.1.1] that a commutative finite flat group scheme can be embedded into an Abelian variety locally in the base. However, an additional argument is needed to ensure that the image of  $\Phi_{n,R}$  consists of truncated displays.

*Remark 4.3.* If  $k$  is a perfect field of characteristic  $p$ , in view of Lemma 3.5, the functor  $\Phi_{n,k} : (p\text{-div}_n/k) \rightarrow (\text{disp}_n/k)$  is an equivalence of categories by classical Dieudonné theory.

*Remark 4.4.* The definition of the dual display carries over to truncated displays, and the functor  $\Phi_{n,R}$  preserves duality because this holds for  $\Phi_R$ ; see Remark 2.3. We leave out the details.

**4.2. Smoothness.** The functors  $\Phi_{n,R}$  for variable  $\mathbb{F}_p$ -algebras  $R$  induce a morphism of algebraic stacks over  $\mathbb{F}_p$

$$\phi_n : \overline{\mathcal{BT}}_n \rightarrow \mathcal{Disp}_n.$$

The source and target of  $\phi_n$  are smooth over  $\mathbb{F}_p$  of pure dimension zero. For a perfect field  $k$  of characteristic  $p$  the functor

$$\phi_n(k) : \overline{\mathcal{BT}}_n(k) \rightarrow \mathcal{Disp}_n(k)$$

is an equivalence; see Remark 4.3.

**Theorem 4.5.** *The morphism  $\phi_n$  is smooth and surjective.*

*Proof.* Let  $\mathcal{U} \subset \overline{\mathcal{BT}}_n$  be the open substack where  $\phi_n$  is smooth. We consider a geometric point in  $\overline{\mathcal{BT}}_n(k)$  for an algebraically closed field  $k$ , given by a truncated Barsotti-Tate group  $G$  over  $k$ . The tangent space  $t_G(\overline{\mathcal{BT}}_n)$  is the set of isomorphism classes of deformations of  $G$  over  $k[\varepsilon]$ . Let  $\mathcal{P} = \phi(G)$ .

Since  $\overline{\mathcal{BT}}_n$  and  $\mathcal{Disp}_n$  are smooth,  $G$  lies in  $\mathcal{U}(k)$  if and only if  $\phi_n$  induces a surjective map on tangent spaces

$$t_G(\phi_n) : t_G(\overline{\mathcal{BT}}_n) \rightarrow t_{\mathcal{P}}(\mathcal{Disp}_n).$$

There is a  $p$ -divisible group  $H$  over  $k$  such that  $G \cong H[p^n]$ . Let  $\tilde{\mathcal{P}} = \Phi_k(H)$  be the associated display, thus  $\mathcal{P} = \tau_n \tilde{\mathcal{P}}$ . We have a commutative square of tangent spaces

$$\begin{array}{ccc} t_H(\overline{\mathcal{BT}}) & \xrightarrow{t_H(\phi)} & t_{\tilde{\mathcal{P}}}(\mathcal{Disp}) \\ t_H(\tau) \downarrow & & \downarrow t_{\tilde{\mathcal{P}}}(\tau) \\ t_G(\overline{\mathcal{BT}}_n) & \xrightarrow{t_G(\phi_n)} & t_{\mathcal{P}}(\mathcal{Disp}_n) \end{array}$$

where  $\tau$  denotes the truncation morphisms and where  $\phi : \overline{\mathcal{BT}} \rightarrow \mathcal{Disp}$  is induced by the functors  $\Phi_R$  for  $\mathbb{F}_p$ -algebras  $R$ . Here  $t_{\tilde{\mathcal{P}}}(\tau)$  is surjective because the truncation morphisms  $\mathcal{Disp}_{m+1} \rightarrow \mathcal{Disp}_m$  for  $m \geq n$  are smooth.

If  $G$  is infinitesimal or unipotent,  $H$  is infinitesimal or unipotent as well, and the map  $t_H(\phi)$  is bijective by Corollary 2.8. Thus  $\mathcal{U}$  contains all infinitesimal and unipotent groups, and  $\mathcal{U} = \overline{\mathcal{BT}}_n$  by Lemma 4.6 below.  $\square$

**Lemma 4.6.** *Let  $\mathcal{U}$  be an open substack of  $\overline{\mathcal{BT}}_n$  that contains all points which correspond to infinitesimal or unipotent groups. Then  $\mathcal{U} = \overline{\mathcal{BT}}_n$ .*

*Proof.* For an algebraically closed field  $k$  and  $G \in \overline{\mathcal{BT}}_n(k)$  we have to show that  $G$  lies in  $\mathcal{U}(k)$ . We write  $G = H[p^n]$  for a  $p$ -divisible group  $H$  over  $k$ . Let  $K$  be an algebraic closure of  $k((t))$  and let  $R$  be the ring of integers of  $K$ . Let  $\nu$  be the Newton polygon of  $H$  and let  $\beta$  be the unique linear Newton polygon with  $\beta \preceq \nu$ . By [O1, Thm. 3.2] there is a  $p$ -divisible group  $H''$  over  $R$  with generic Newton polygon  $\nu$  and special Newton polygon  $\beta$ . Since  $K$  is algebraically closed, there is an isogeny  $H'_K \rightarrow H \otimes_k K$ . Let  $C$  be its kernel, let  $C_R \subset H''$  be the schematic closure of  $C$ , and let  $H' = H''/C_R$ . Then  $H'_K \cong H \otimes_k K$ , and the special fibre  $H'_k$  is isoclinic. We obtain a commutative diagram where  $g$  is given by  $G$ , and  $g'$  is given by  $H'[p^n]$ .

$$\begin{array}{ccc} \mathrm{Spec} K & \longrightarrow & \mathrm{Spec} k \\ \downarrow & & \downarrow g \\ \mathrm{Spec} R & \xrightarrow{g'} & \overline{\mathcal{BT}}_n. \end{array}$$

Here  $g'^{-1}(\mathcal{U})$  is an open subset of  $\mathrm{Spec} R$ , which contains the closed point since the special fibre of  $H'[p^n]$  is unipotent or infinitesimal. Thus  $g'^{-1}(\mathcal{U})$  is all of  $\mathrm{Spec} R$ , which implies that  $G$  lies in  $\mathcal{U}(k)$ .  $\square$

We consider the diagonal morphism

$$\Delta : \overline{\mathcal{BT}}_n \rightarrow \overline{\mathcal{BT}}_n \times_{\mathcal{Disp}_n} \overline{\mathcal{BT}}_n$$

and view it as a morphism over  $\overline{\mathcal{BT}}_n \times \overline{\mathcal{BT}}_n$ . Let  $X$  be an affine  $\mathbb{F}_p$ -scheme. For  $g : X \rightarrow \overline{\mathcal{BT}}_n \times \overline{\mathcal{BT}}_n$ , corresponding to two truncated Barsotti-Tate groups  $G_1$  and  $G_2$  over  $X$ , with associated truncated displays  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the inverse image of  $\Delta$  under  $g$  is the morphism of affine  $X$ -schemes

$$\mathrm{Isom}(G_1, G_2) \rightarrow \mathrm{Isom}(\mathcal{P}_1, \mathcal{P}_2).$$

For  $G \in \overline{\mathcal{BT}}_n(X)$ , with associated truncated display  $\mathcal{P}$ , let

$$\underline{\mathrm{Aut}}^o(G) = \mathrm{Ker}(\underline{\mathrm{Aut}} G \rightarrow \underline{\mathrm{Aut}} \mathcal{P}).$$

This is an affine group scheme over  $X$ . For varying  $X$  and  $G$  we obtain a relative affine group scheme  $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  over  $\overline{\mathcal{BT}}_n$ . Let

$$\pi_1, \pi_2 : \overline{\mathcal{BT}}_n \times_{\mathcal{D}isp_n} \overline{\mathcal{BT}}_n \rightarrow \overline{\mathcal{BT}}_n$$

be the two projections.

**Theorem 4.7.** *The representable affine morphism  $\Delta$  is finite, flat, radicial, and surjective. The group scheme  $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}}) \rightarrow \overline{\mathcal{BT}}_n$  is commutative and finite flat, and  $\Delta$  is a torsor under  $\pi_i^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  for  $i = 1, 2$ .*

*Proof.* We write  $\mathcal{X} = \overline{\mathcal{BT}}_n$  and  $\mathcal{Y} = \mathcal{D}isp_n$ . Let  $\pi : X \rightarrow \mathcal{X}$  be a smooth presentation with affine  $X$ . We can assume that  $X$  has pure dimension  $m$ , which implies that  $\pi$  has pure dimension  $m$ . By Theorem 4.5, the composition  $\psi = \phi_n \circ \pi : X \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$  is a smooth presentation of pure dimension  $m$  as well. It follows that  $X' = X \times_{\mathcal{X}} X$  and  $Y' = X \times_{\mathcal{Y}} X$  are smooth  $\mathbb{F}_p$ -schemes of pure dimension  $2m$ . The natural morphism  $\phi' : X' \rightarrow Y'$  can be identified with the inverse image of  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$  under the smooth presentation  $X \times X \rightarrow \mathcal{X} \times \mathcal{X}$ .

Since  $\phi_n : \mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence on geometric points,  $\phi'$  is bijective on geometric points. Since  $X'$  and  $Y'$  are equidimensional, the irreducible components of  $X'$  are in bijection to the irreducible components of  $Y'$ . Thus  $\phi'$  is flat; see [Ma, Thm. 23.1]. Let  $Z$  be the normalisation of  $Y'$  in the purely inseparable extension of function fields defined by  $X' \rightarrow Y'$ . Then  $Z \rightarrow Y'$  is bijective on geometric points, so  $X' \rightarrow Z$  is bijective on geometric points, and  $X' = Z$  by Zariski's main theorem. Thus  $\phi'$  is finite, flat, radicial, and surjective, which implies that  $\Delta$  is finite, flat, radicial, and surjective.

Recall that a morphism  $T \rightarrow S$  with an action of an  $S$ -group  $A$  on  $T$  is called a quasi-torsor if for each  $S' \rightarrow S$  the fibre  $T(S')$  is either empty or isomorphic to  $A(S')$  as an  $A(S')$ -set. Clearly  $\Delta$  is a quasi-torsor under the obvious right action of  $\pi_1^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  and under the obvious left action of  $\pi_2^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ . Since  $\Delta$  is finite, flat, and surjective, it follows that the quasi-torsor  $\Delta$  is a torsor, and that  $\pi_i^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  is finite and flat over  $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ . Since  $\phi_n$  is smooth and surjective, the same holds for the projections  $\pi_i$ , and it follows that  $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  is finite and flat over  $\mathcal{X}$ .

It remains to show that  $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  is commutative. It suffices to show this on a dense open substack of  $\mathcal{X}$ , and thus it suffices to show that for  $G = (\mathbb{Z}/p^n\mathbb{Z})^r \times (\mu_{p^n})^s$  over a field  $k$  the  $k$ -group scheme  $\underline{\mathrm{Aut}}^o(G)$  is commutative. Now  $\underline{\mathrm{Hom}}(\mu_{p^n}, \mathbb{Z}/p^n\mathbb{Z})$  is zero, and the group schemes  $\underline{\mathrm{Aut}}((\mathbb{Z}/p^n\mathbb{Z})^r)$  and  $\underline{\mathrm{Aut}}((\mu_{p^n})^s)$  are étale. Since  $\phi_n$  is an equivalence on geometric points, it follows that  $\underline{\mathrm{Aut}}^o(G)$  is contained in the group scheme  $\{(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}) \mid a \in \mu_{p^n}^{rs}\}$ , which is commutative.  $\square$

*Remark 4.8.* For  $G = (\mathbb{Z}/p^n\mathbb{Z})^r \times (\mu_{p^n})^s$  as above,  $\underline{\mathrm{Aut}}^o(G)$  is in fact equal to  $\{(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}) \mid a \in \mu_{p^n}^{rs}\}$ . Since ordinary groups are dense in  $\overline{\mathcal{BT}}_n$ , it follows that on the open and closed substack of  $\overline{\mathcal{BT}}_n$  where the universal group has dimension  $s$  and codimension  $r$ , the degree of the finite flat group scheme  $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  is equal to  $p^{rsn}$ .

To prove the first equality, it suffices to show that the truncated displays  $\mathcal{P}_1 = \Phi_{n, \mathbb{F}_p}(\mathbb{Z}/p^n\mathbb{Z})$  and  $\mathcal{P}_2 = \Phi_{n, \mathbb{F}_p}(\mu_{p^n})$  satisfy  $\underline{\mathrm{Hom}}(\mathcal{P}_1, \mathcal{P}_2) = 0$ . For an  $\mathbb{F}_p$ -algebra  $R$ , if  $i : I_{n+1, R} \rightarrow W_n(R)$  denotes the natural homomorphism, we have

$$\begin{aligned}\mathcal{P}_1 &= (W_n(R), W_n(R), \mathrm{id}, i, f, pf), \\ \mathcal{P}_2 &= (W_n(R), I_{n+1, R}, i, \mathrm{id}, f_1, f).\end{aligned}$$

Thus  $\underline{\mathrm{Hom}}(\mathcal{P}_1, \mathcal{P}_2)$  can be identified with the set of  $a \in I_{n+1, R}$  such that  $f_1(a) = i(a)$ , or equivalently  $a = v(i(a))$ , which implies that  $a = 0$ .

By passing to the limit, Theorems 4.5 and 4.7 give the following information on the morphism  $\phi : \overline{\mathcal{BT}} \rightarrow \mathcal{Disp}$ . For a  $p$ -divisible group  $G$  over an  $\mathbb{F}_p$ -algebra  $R$  with associated display  $\mathcal{P}$  let  $\underline{\mathrm{Aut}}^o(G)$  be the kernel of  $\underline{\mathrm{Aut}}(G) \rightarrow \underline{\mathrm{Aut}}(\mathcal{P})$ . This is an affine group scheme over  $R$ , which is the projective limit over  $n$  of the finite flat group schemes  $\underline{\mathrm{Aut}}^o(G[p^n])$ ; thus  $\underline{\mathrm{Aut}}^o(G)$  is commutative and flat. Let  $G^{\mathrm{univ}}$  be the universal  $p$ -divisible group and let  $\pi_1, \pi_2 : \mathcal{Disp} \times_{\overline{\mathcal{BT}}} \mathcal{Disp} \rightarrow \mathcal{Disp}$  be the two projections.

**Corollary 4.9.** *The morphism  $\phi$  is faithfully flat, and its diagonal is a torsor under the flat affine group scheme  $\pi_i^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$  for  $i = 1, 2$ .  $\square$*

The limit  $\underline{\mathrm{Aut}}^o(G) = \varprojlim_n \underline{\mathrm{Aut}}^o(G[p^n])$  can show quite different behaviour depending on  $G$ ; see Corollary 5.6 in the next section.

## 5. CLASSIFICATION OF FORMAL $p$ -DIVISIBLE GROUPS

As an application of Theorems 4.5 and 4.7 together with Corollary 2.8 we will prove the following.

**Theorem 5.1.** *For each  $p$ -adic ring  $R$ , the functor  $\Phi_R$  from  $p$ -divisible group over  $R$  to displays over  $R$  induces an equivalence*

$$\Phi_R^1 : (\text{formal } p\text{-divisible groups}/R) \rightarrow (\text{nilpotent displays}/R).$$

It is known by [Zi1] and [La1] that the functor  $\mathrm{BT}_R$  defined in [Zi1] from displays over  $R$  to formal Lie groups over  $R$  induces an equivalence between nilpotent displays over  $R$  and formal  $p$ -divisible groups over  $R$ . The relation between  $\Phi_R$  and  $\mathrm{BT}_R$  is discussed in section 8.

**Lemma 5.2.** *For  $n \geq 1$  there is a Cartesian diagram of Artin algebraic stacks*

$$\begin{array}{ccc} \mathcal{BT}_n^o & \xrightarrow{\phi_n^o} & \mathcal{Disp}_n^o \\ \downarrow & & \downarrow \\ \overline{\mathcal{BT}}_n & \xrightarrow{\phi_n} & \mathcal{Disp}_n \end{array}$$

where the vertical arrows are the immersions given by Lemmas 1.9 and 3.17, and where  $\phi_n$  is given by the functors  $\Phi_{n, R}$ . The projective limit over  $n$  is a Cartesian diagram of affine algebraic stacks

$$\begin{array}{ccc} \mathcal{BT}^o & \xrightarrow{\phi^o} & \mathcal{Disp}^o \\ \downarrow & & \downarrow \\ \overline{\mathcal{BT}} & \xrightarrow{\phi} & \mathcal{Disp} \end{array}$$

where  $\phi$  is given by the functors  $\Phi_R$ .

*Proof.* Since  $\phi_n$  is smooth, the inverse image of  $\mathcal{D}isp_n^o$  under  $\phi_n$  is a reduced closed substack  $\mathcal{BT}'_n$  of  $\overline{\mathcal{BT}}_n$ . By classical Dieudonné theory, the geometric points of  $\mathcal{BT}'_n$  and of  $\mathcal{BT}_n^o$  coincide; thus  $\mathcal{BT}'_n = \mathcal{BT}_n^o$ , and we get the first Cartesian square. The projective limit of  $\mathcal{BT}'_n$  is  $\mathcal{BT}^o$  by Lemma 1.9, and the projective limit of  $\mathcal{D}isp_n^o$  is  $\mathcal{D}isp^o$  by Lemma 3.17. Hence the second Cartesian square follows from the first one.  $\square$

The essential part of Theorem 5.1 is the following result.

**Theorem 5.3.** *The morphism  $\phi^o : \mathcal{BT}^o \rightarrow \mathcal{D}isp^o$  is an isomorphism.*

*Proof.* We use the following notation.

$$\begin{aligned} \mathcal{X} &= \mathcal{BT}^o, & \mathcal{X}_n &= \mathcal{BT}_n^o, \\ \mathcal{Y} &= \mathcal{D}isp^o, & \mathcal{Y}_n &= \mathcal{D}isp_n^o. \end{aligned}$$

As in the proof of Lemma 1.6 we choose smooth presentations  $X_n \rightarrow \mathcal{X}_n$  with affine  $X_n$  such that the truncation morphisms  $\mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$  lift to morphisms  $X_{n+1} \rightarrow X_n$  where  $X_{n+1} \rightarrow X_n \times_{\mathcal{X}_n} \mathcal{X}_{n+1}$  is smooth and surjective. Since  $\mathcal{X}_n \rightarrow \mathcal{Y}_n$  is smooth and surjective by Theorem 4.5 and Lemma 5.2, the composition  $X_n \rightarrow \mathcal{X}_n \rightarrow \mathcal{Y}_n$  is a smooth presentation. Let  $X = \varprojlim_n X_n$  and

$$\begin{aligned} X'_n &= X_n \times_{\mathcal{X}_n} X_n, & X' &= X \times_{\mathcal{X}} X = \varprojlim_n X'_n, \\ Y'_n &= X_n \times_{\mathcal{Y}_n} X_n, & Y' &= X \times_{\mathcal{Y}} X = \varprojlim_n Y'_n. \end{aligned}$$

Here  $X \rightarrow \mathcal{X}$  is a faithfully flat presentation because for  $Z \rightarrow \mathcal{X}$  with affine  $Z$  we have  $Z \times_{\mathcal{X}} X = \varprojlim_n (Z \times_{\mathcal{X}_n} X_n)$ , and a projective limit of faithfully flat affine  $Z$ -schemes is a faithfully flat affine  $Z$ -scheme. Similarly, the composition  $X \rightarrow \mathcal{X} \rightarrow \mathcal{Y}$  is a faithfully flat presentation. We have an infinite commutative diagram:

$$\begin{array}{ccccccc} X'_1 & \longleftarrow & X'_2 & \longleftarrow & X'_3 & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y'_1 & \longleftarrow & Y'_2 & \longleftarrow & Y'_3 & \longleftarrow & \cdots \end{array}$$

The theorem means that the limit  $X' \rightarrow Y'$  is an isomorphism.

Let  $G_n$  be the infinitesimal truncated Barsotti-Tate group over  $X_n$  which defines the presentation  $X_n \rightarrow \mathcal{X}_n$  and let  $\pi_n : Y'_n \rightarrow X_n$  be the first projection. By Theorem 4.7,  $X'_n \rightarrow Y'_n$  is a torsor under the commutative infinitesimal finite flat group scheme  $A_n = \underline{\text{Aut}}^o(\pi_n^* G_n)$ . The truncation induces a homomorphism of finite flat group schemes over  $Y'_{n+1}$

$$\psi_n : A_{n+1} \rightarrow A_n \times_{Y'_n} Y'_{n+1}$$

and a morphism

$$X'_{n+1} \rightarrow X'_n \times_{Y'_n} Y'_{n+1}$$

which is equivariant with respect to  $\psi_n$ .



**Lemma 5.4.** *For each  $m$  there is an  $n \geq m$  such that the transition homomorphism*

$$\psi_{m,n} : A_n \rightarrow A_m \times_{Y'_m} Y'_n$$

*is zero.*

If Lemma 5.4 is proved, it follows that there is a unique diagonal morphism which makes the following diagram commute.

$$\begin{array}{ccc} X'_m & \longleftarrow & X'_n \\ \downarrow & \swarrow & \downarrow \\ Y'_m & \longleftarrow & Y'_n \end{array}$$

Thus  $\varprojlim_n X'_n \rightarrow \varprojlim_n Y'_n$  is an isomorphism, and Theorem 5.3 follows.

The essential case of Lemma 5.4 is the following. Consider a geometric point  $y : \operatorname{Spec} k \rightarrow Y'$  for an algebraically closed field  $k$ . Let  $A_{n,k} = A_n \times_{Y'_n} \operatorname{Spec} k$  and  $Z_n = X'_n \times_{Y'_n} \operatorname{Spec} k$ . Thus  $Z_n$  is an  $A_{n,k}$ -torsor. We have homomorphisms of finite  $k$ -group schemes  $A_{n+1,k} \rightarrow A_{n,k}$  and equivariant morphisms  $Z_{n+1} \rightarrow Z_n$ . Since  $k$  is algebraically closed and since  $A_{n,k}$  is infinitesimal,  $Z_n(k)$  has precisely one element, and there are compatible isomorphisms  $Z_n \cong A_{n,k}$ . For each  $m$  the images of  $A_{n,k} \rightarrow A_{m,k}$  for  $n \geq m$  stabilise to a subgroup scheme  $A'_m$  of  $A_{m,k}$ , and  $A'_{m+1} \rightarrow A'_m$  is an epimorphism. Let  $A'_n = \operatorname{Spec} B_n$  and  $B = \varinjlim_n B_n$ .

The geometric point  $y : \operatorname{Spec} k \rightarrow Y'$  corresponds to two formal  $p$ -divisible groups  $G_1$  and  $G_2$  over  $k$  together with an isomorphism of the associated displays  $\alpha : \Phi_k(G_1) \cong \Phi_k(G_2)$ . The  $k$ -scheme  $X' \times_{Y'} \operatorname{Spec} k$  can be identified with  $\varprojlim_n Z_n = \varprojlim_n A'_n = \operatorname{Spec} B$  and classifies lifts of  $\alpha$  to an isomorphism  $G_1 \cong G_2$ . Thus Corollary 2.8 implies that  $B$  is a formally étale  $k$ -algebra. It follows that the cotangent complex  $L_{B/k}$  has trivial homology in degrees 0 and 1; see Lemma 5.5 below. The epimorphism  $A'_{n+1} \rightarrow A'_n$  induces an injective homomorphism  $H_1(L_{B_n/k}) \otimes_{B_n} B_{n+1} \rightarrow H_1(L_{B_{n+1}/k})$ ; see [Me1, Chap. I, Prop. 3.3.4]. Since  $L_{B/k} = \varinjlim_n L_{B_n/k}$  it follows that  $H_1(L_{B_n/k})$  is zero, which implies that the finite infinitesimal group scheme  $A'_n$  is zero for each  $n$ . Thus  $X' \times_{Y'} \operatorname{Spec} k \cong \operatorname{Spec} k$ .

Let us now prove Lemma 5.4. First we note that  $X'_{n+1} \rightarrow X'_n$  is smooth and surjective because this morphism can be factored as follows.

$$X_{n+1} \times_{\mathcal{X}_{n+1}} X_{n+1} \rightarrow (X_n \times_{\mathcal{X}_n} \mathcal{X}_{n+1}) \times_{\mathcal{X}_{n+1}} (X_n \times_{\mathcal{X}_n} \mathcal{X}_{n+1}) \rightarrow X_n \times_{\mathcal{X}_n} X_n.$$

The first arrow is smooth and surjective by our assumptions on  $X_n$ , and the second arrow is smooth and surjective because this holds for  $\mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$  by Lemma 1.9. Since  $X'_n \rightarrow Y'_n$  is faithfully flat, it follows that  $Y'_{n+1} \rightarrow Y'_n$  is faithfully flat as well.

Let  $U_{m,n} \subset Y'_n$  be the open set of all points  $y$  such that the fibre  $(\psi_{m,n})_y$  is non-zero. Since  $Y'_n$  is reduced, it suffices to show that for each  $m$  there is an  $n$  such that  $U_{m,n}$  is empty. Assume that for some  $m$  the set  $U_{m,n}$  is non-empty for all  $n \geq m$ . Let  $V_{m,n}$  be the set of generic points of  $U_{m,n}$ . Since  $Y'_{n+1} \rightarrow Y'_n$  is flat, we have  $V_{m,n+1} \rightarrow V_{m,n}$ . Since  $V_{m,n}$  is finite and non-empty, the projective limit over  $n$  of  $V_{m,n}$  is non empty. Hence there is a geometric point  $y : \operatorname{Spec} k \rightarrow Y'$  such that  $\operatorname{Spec} k \rightarrow Y'_n$  lies in  $V_{m,n}$

for each  $n \geq m$ . If we perform the above construction for  $y$ , the group  $A'_m$  is non-zero, which is impossible. Thus Lemma 5.4 and Theorem 5.3 are proved.  $\square$

**Lemma 5.5.** *A ring homomorphism  $\alpha : A \rightarrow B$  is formally étale if and only if the cotangent complex  $L_{B/A}$  has trivial homology in degrees 0 and 1.*

*Proof.* Clearly  $\alpha$  is formally unramified if and only if  $\Omega_{B/A} = H_0(L_{B/A})$  is zero. Let us assume that this holds. Since the obstructions to formal smoothness lie in  $\text{Ext}_B^1(L_{B/A}, M)$  for varying  $B$ -modules  $M$ , the implication  $\Leftarrow$  of the lemma is clear. So assume that  $\alpha$  is formally smooth. We write  $B = R/I$  for a polynomial ring  $R$  over  $A$ ; let  $C = R/I^2$ . Since  $L_{R/A}$  is a free  $R$ -module in degree zero, the natural homomorphisms  $H_1(L_{C/A}) \rightarrow H_1(L_{C/R})$  and  $H_1(L_{B/A}) \rightarrow H_1(L_{B/R})$  are injective. The homomorphism  $H_1(L_{C/R}) \rightarrow H_1(L_{B/R})$  can be identified with  $I^2/I^4 \rightarrow I/I^2$ , which is zero. Thus  $u : H_1(L_{C/A}) \rightarrow H_1(L_{B/A})$  is zero as well. Since  $\alpha$  is formally smooth,  $\text{id}_B$  factors into  $A$ -algebra homomorphisms  $B \rightarrow C \xrightarrow{\pi} B$ , where  $\pi$  is the projection. Thus  $u$  is surjective, and  $H_1(L_{B/A})$  is zero.  $\square$

*Proof of Theorem 5.1.* We may assume that  $p$  is nilpotent in  $R$ . Since  $\Phi_R$  is an additive functor, in order to show that  $\Phi_R^1$  is fully faithful it suffices to show that for two formal  $p$ -divisible groups  $G_1$  and  $G_2$  over  $R$  with associated nilpotent displays  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the map  $\gamma : \text{Isom}(G_1, G_2) \rightarrow \text{Isom}(\mathcal{P}_1, \mathcal{P}_2)$  is bijective. We define an ideal  $I \subset R$  by the Cartesian diagram

$$\begin{array}{ccc} \text{Spec } R/I & \longrightarrow & \mathcal{BT}^o \times \mathcal{BT}^o \\ \downarrow & & \downarrow \\ \text{Spec } R & \xrightarrow{(G_1, G_2)} & \mathcal{BT} \times \mathcal{BT} \end{array}$$

If we write  $G'_1 = G_1 \otimes_R R/I$  etc.,  $\text{Isom}(G'_1, G'_2) \rightarrow \text{Isom}(\mathcal{P}'_1, \mathcal{P}'_2)$  is bijective by Theorem 5.3. Since  $\mathcal{BT}^o \rightarrow \mathcal{BT}$  is of finite presentation by Lemma 1.9 and since  $I$  is a nil-ideal,  $I$  is nilpotent. By Corollary 2.8 it follows that  $\gamma$  is bijective. We show that  $\Phi_R^1$  is essentially surjective. Since  $\mathcal{D}isp^o \rightarrow \mathcal{D}isp$  is of finite presentation by Lemma 3.17, for a given nilpotent display  $\mathcal{P}$  over  $R$  we find a nilpotent ideal  $I \subset R$  such that the associated morphism  $\text{Spec } R/I \rightarrow \text{Spec } R \rightarrow \mathcal{D}isp$  factors over  $\mathcal{D}isp^o$ . By Theorem 5.3,  $\mathcal{P}_{R/I}$  lies in the image of  $\Phi_{R/I}^1$ . Thus  $\mathcal{P}$  lies in the image of  $\Phi_R^1$  by Corollary 2.8.  $\square$

**Corollary 5.6.** *Let  $G$  be a  $p$ -divisible group over an  $\mathbb{F}_p$ -algebra  $R$ . The affine group scheme  $\underline{\text{Aut}}^o(G)$  is trivial if and only if for all  $x \in \text{Spec } R$  the fibre  $G_x$  is connected or unipotent.*

*Proof.* Assume that  $G$  is a  $p$ -divisible group over an algebraically closed field  $k$  such that  $H = \mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$  is a direct summand of  $G$ . By Remark 4.8, the group  $\underline{\text{Aut}}^o(H[p^n])$  is isomorphic to  $\mu_{p^n}$ , with transition maps  $\mu_{p^{n+1}} \rightarrow \mu_{p^n}$  given by  $\zeta \mapsto \zeta^p$ . Thus  $\underline{\text{Aut}}^o(H)$  is non-trivial, which implies that  $\underline{\text{Aut}}^o(G)$  is non-trivial as well. This proves the implication  $\Rightarrow$ .

Assume now that all fibres of  $G$  over  $R$  are connected or unipotent. Let  $\text{Spec } R/I$  and  $\text{Spec } R/J$  be the inverse images of  $\mathcal{BT}^o$  under the morphisms  $\text{Spec } R \rightarrow \mathcal{BT}$  defined by  $G$  and by  $G^\vee$ . Then  $I$  and  $J$  are finitely generated

by Lemma 1.9, and  $IJ$  is a nilideal, thus  $IJ$  is nilpotent. In order to show that  $\underline{\text{Aut}}^o(G)$  is trivial we may replace  $R$  by  $R/IJ$ . Then  $R \rightarrow R/I \times R/J$  is injective. Since  $\underline{\text{Aut}}^o(G)$  is flat over  $R$  by Corollary 4.9, we may further replace  $R$  by  $R/I$  and by  $R/J$ . Since  $\underline{\text{Aut}}^o(G) \cong \underline{\text{Aut}}^o(G^\vee)$  by Remark 2.3, in both cases the assertion follows from Theorem 5.3.  $\square$

## 6. DIEUDONNÉ THEORY OVER PERFECT RINGS

The results of this section were obtained earlier by Gabber by a different method. For a perfect ring  $R$  of characteristic  $p$  we consider the Dieudonné ring

$$D(R) = W(R)\{F, V\}/J$$

where  $W(R)\{F, V\}$  is the non-commutative polynomial ring in two variables over  $W(R)$  and where  $J$  is the ideal generated by the relations  $Fa = f(a)F$  and  $aV = Vf(a)$  for  $a \in W(R)$ , and  $FV = VF = p$ .

**Definition 6.1.** A *projective Dieudonné module* over  $R$  is a  $D(R)$ -module which is a projective  $W(R)$ -module of finite type. A *truncated Dieudonné module of level  $n$*  over  $R$  is a  $D(R)$ -module  $M$  which is a projective  $W_n(R)$ -module of finite type; if  $n = 1$  we require also that  $\text{Ker}(F) = \text{Im}(V)$  and  $\text{Ker}(V) = \text{Im}(F)$  and that  $M/VM$  is a projective  $R$ -module. An *admissible torsion  $W(R)$ -module* is a finitely presented  $W(R)$ -module of projective dimension  $\leq 1$  which is annihilated by a power of  $p$ . A *finite Dieudonné module* over  $R$  is a  $D(R)$ -module which is an admissible torsion  $W(R)$ -module.

We denote by  $(\text{Dieud}/R)$ , by  $(\text{Dieud}_n/R)$ , and by  $(\text{Dieud}^f/R)$  the categories of projective, truncated level  $n$ , and finite Dieudonné modules over  $R$ , respectively. For a homomorphism of perfect rings  $R \rightarrow R'$  the scalar extension by  $W(R) \rightarrow W(R')$  induces functors from projective, truncated, or finite Dieudonné modules over  $R$  to such modules over  $R'$ .

**Lemma 6.2.** *If  $M$  is a projective or truncated Dieudonné module of level  $\geq 2$  then  $M/pM$  is a truncated Dieudonné module of level 1.*

*Proof.* We can replace  $M$  by  $M/p^2M$ . The operators  $F$  and  $V$  of  $M$  induce operators  $\bar{F}$  and  $\bar{V}$  of  $\bar{M} = M/pM$ . Assume that  $\bar{F}(\bar{x}) = 0$  for an element  $x \in M$ . Then  $F(x) \in pM$  and thus  $F(x) = F(V(y))$  for some  $y \in M$ . Hence  $px = pV(y)$  and thus  $\bar{x} = \bar{V}(\bar{y})$ . This shows that  $\text{Ker}(\bar{F}) = \text{Im}(\bar{V})$ , and similarly  $\text{Ker}(\bar{V}) = \text{Im}(\bar{F})$ . Since these relations remain true after base change, for each point of  $\text{Spec } R$  the dimensions of the fibres of  $M/VM$  and of  $M/FM$  add up to the dimension of the fibre of  $\bar{M}$ . Thus the dimension of the fibre of  $M/VM$  is an upper and lower semicontinuous function on  $\text{Spec } R$ , and  $M/VM$  is a projective  $R$ -module since  $R$  is reduced.  $\square$

**Lemma 6.3.** *Truncated Dieudonné modules of level  $n \geq 1$  over  $R$  are equivalent to truncated displays of level  $n$  over  $R$ , and projective Dieudonné modules over  $R$  are equivalent to displays over  $R$ .*

*Proof.* This extends Lemma 3.5. Since multiplication by  $p$  is an isomorphism  $W_n(R) \cong I_{n+1,R}$  and since truncated displays have normal decompositions by Lemma 3.3, truncated displays of level  $n$  over  $R$  are equivalent to quintuples  $\mathcal{P} = (P, Q, \iota, \epsilon, F_1)$  where  $P$  and  $Q$  are projective  $W_n(R)$ -modules of

finite type with homomorphisms  $P \xrightarrow{\epsilon} Q \xrightarrow{\iota} P$  such that  $\epsilon\iota = p$  and  $\iota\epsilon = p$ , and where  $F_1 : Q \rightarrow P$  is a bijective  $f$ -linear map, such that

( $\star$ )  $\text{Coker}(\iota)$  is projective over  $R$ , and  $Q \xrightarrow{\iota} P \xrightarrow{p^{n-1}\epsilon} Q \xrightarrow{\iota} P$  is exact.

By the proof of Lemma 6.2, condition ( $\star$ ) is automatic if  $n \geq 2$ . It follows that truncated displays of level  $n \geq 1$  are equivalent to truncated Dieudonné modules of level  $n$  by  $\mathcal{P} \mapsto (P, F, V)$  with  $F = F_1\epsilon$  and  $V = \iota F_1^{-1}$ . The equivalence between displays and projective Dieudonné modules follows easily; see also [La3, Lemma 2.4].  $\square$

**Theorem 6.4.** *For a perfect ring  $R$  of characteristic  $p$ , the functors*

$$\begin{aligned}\Phi_{n,R} &: (p\text{-div}_n/R) \rightarrow (\text{Dieud}_n/R), \\ \Phi_R &: (p\text{-div}/R) \rightarrow (\text{Dieud}/R)\end{aligned}$$

*are equivalences; these functors are well-defined by Lemma 6.3.*

Here  $\Phi_R$  and  $\Phi_{n,R}$  are defined by  $G \mapsto (\mathbb{D}(G)_{W(R)/R}, F, V)$ , where  $\mathbb{D}(G)$  is the covariant Dieudonné crystal of  $G$ , and where  $F$  and  $V$  are induced by the Verschiebung  $G^{(1)} \rightarrow G$  and Frobenius  $G \rightarrow G^{(1)}$ .

*Proof.* For an  $\mathbb{F}_p$ -algebra  $A$ , the perfection  $A^{\text{per}}$  is the direct limit of Frobenius  $A \rightarrow A \rightarrow \dots$ . For an  $\mathbb{F}_p$ -scheme  $X$ , the perfection  $X^{\text{per}}$  is the projective limit of Frobenius  $X \leftarrow X \leftarrow \dots$ . This is a local construction, which coincides with the perfection of rings in the case of affine schemes.

Since the functor  $\Phi_{n,R}$  is additive, it is fully faithful if for two groups  $G_1$  and  $G_2$  in  $(p\text{-div}_n/R)$  with associated truncated displays  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the map  $\gamma : \text{Isom}(G_1, G_2) \rightarrow \text{Isom}(\mathcal{P}_1, \mathcal{P}_2)$  induced by  $\Phi_{n,R}$  is bijective. The morphism of affine  $R$ -schemes  $\underline{\text{Isom}}(G_1, G_2) \rightarrow \underline{\text{Isom}}(\mathcal{P}_1, \mathcal{P}_2)$  is a torsor under an infinitesimal finite flat group scheme by Theorem 4.7. Thus  $\underline{\text{Isom}}(G_1, G_2)^{\text{per}} \rightarrow \underline{\text{Isom}}(\mathcal{P}_1, \mathcal{P}_2)^{\text{per}}$  is an isomorphism, which implies that  $\gamma$  is bijective.

Let us show that  $\Phi_{n,R}$  is essentially surjective. If  $X \rightarrow \mathcal{BT}_n$  is a smooth presentation with affine  $X$ , the composition  $X \rightarrow \mathcal{BT}_n \rightarrow \mathcal{D}isp_n$  is smooth and surjective by Theorem 4.5. For a given truncated display  $\mathcal{P}$  of level  $n$  over  $R$  let  $\text{Spec } S = X \times_{\mathcal{D}isp_n} \text{Spec } R$  and let  $R' = S^{\text{per}}$ . Then  $R'$  is a perfect faithfully flat  $R$ -algebra such that  $\mathcal{P}_{R'}$  lies in the image of  $\Phi_{n,R'}$ . Since  $R'' = R' \otimes_R R'$  is perfect, the functors  $\Phi_{n,R'}$  and  $\Phi_{n,R''}$  are fully faithful. Thus  $\mathcal{P}$  lies in the image of  $\Phi_{n,R}$  by faithfully flat descent.

By passing to the projective limit it follows that  $\Phi_R$  is an equivalence.  $\square$

We denote by  $(p\text{-grp}/R)$  the category of commutative finite flat group schemes of  $p$ -power order over  $R$ .

**Corollary 6.5.** *The covariant Dieudonné crystal defines a functor*

$$\Phi_R^f : (p\text{-grp}/R) \rightarrow (\text{Dieud}^f/R)$$

*which is an equivalence of categories.*

Over perfect field this is classical, over perfect valuation rings the result is proved in [Be], and the general case was first proved by Gabber by a reduction to the case of valuation rings. Theorem 6.4 is an immediate consequence of Corollary 6.5.

The functor  $\Phi_R^f$  can be defined by  $G \mapsto (\mathbb{D}(G)_{W(R)}, F, V)$  as above.

*Proof of Corollary 6.5.* By a standard construction, the functor  $\Phi_R$  and its inverse  $\Phi_R^{-1}$  induce formally a functor  $\Phi_R^f$  as in Corollary 6.5 and a functor  $\Psi_R^f$  in the opposite direction, which are mutually inverse; this new definition of  $\Phi_R^f$  coincides with the previous one by the construction of  $\Phi_R$ .

Let us explain this in more detail. Since Zariski descent is effective for finite flat group schemes and for finite Dieudonné modules, in order to define  $\Phi_R^f$  and  $\Psi_R^f$  we may always pass to an open cover of  $\text{Spec } R$ . For each group  $G$  in  $(p\text{-grp}/R)$ , by a theorem of Raynaud [BBM, Thm. 3.1.1] there is an open cover of  $\text{Spec } R$  where  $G$  can be written as the kernel of an isogeny of  $p$ -divisible groups  $H_0 \rightarrow H_1$ . We define  $\Phi_R^f(G) = \text{Coker}[\Phi_R(H_0) \rightarrow \Phi_R(H_1)]$ . This is independent of the chosen isogeny, functorial in  $G$ , and compatible with localisations in  $R$ ; see the proof of Proposition 4.1.

A homomorphism of projective Dieudonné modules  $u : N_0 \rightarrow N_1$  over  $R$  is called an isogeny if  $u$  becomes bijective when  $p$  is inverted. Then  $u$  is injective, and its cokernel is a finite Dieudonné module  $M$ . In this case we define  $\Psi_R^f(M) = \text{Ker}[\Phi_R^{-1}(N_0) \rightarrow \Phi_R^{-1}(N_1)]$ . This depends only on  $M$ , the construction is functorial in  $M$ , compatible with localisations in  $R$ , and inverse to  $\Phi_R^f$  when it is defined.

It remains to show that each finite Dieudonné module  $M$  over  $R$  can be written as the cokernel of an isogeny of projective Dieudonné modules locally in  $\text{Spec } R$ . It is easy to find a commutative diagram

$$\begin{array}{ccccc} N_1 & \xrightarrow{\epsilon} & Q & \xrightarrow{\iota} & N_1 \\ \downarrow \pi & & \downarrow \psi & & \downarrow \pi \\ M & \xrightarrow{F} & M & \xrightarrow{V} & M \end{array}$$

where  $Q$  and  $N$  are free  $W(R)$ -modules of the same finite rank, where  $\iota, \epsilon, \pi$  are  $W(R)$ -linear maps, and where  $\psi$  is an  $f$ -linear map, such that  $\iota\epsilon = p$  and  $\epsilon\iota = p$  and  $\pi, \psi$  are surjective. The kernel of  $\pi$  is a projective  $W(R)$ -module  $N_0$  of finite type. If we find a bijective  $f$ -linear map  $F_1 : Q \rightarrow N_1$  with  $\pi F_1 = \psi$ , we can define  $F = F_1\epsilon$  and  $V = \iota F_1^{-1}$ , and  $M$  is the cokernel of the isogeny of Dieudonné modules  $N_0 \rightarrow N_1$ . Thus it suffices to show that  $F_1$  exists locally in  $\text{Spec } R$ , which is a easy application of Nakayama's lemma.  $\square$

## 7. SMALL PRESENTATIONS

In addition to the infinite dimensional presentation of  $\mathcal{BT}$  constructed in Lemma 1.6 one can also find presentations where the covering space is noetherian, or even of finite type. This will be used in section 8. Assume that  $G$  is a  $p$ -divisible group over a  $\mathbb{Z}_p$ -algebra  $A$ . It defines a morphism

$$\pi : \text{Spec } A \rightarrow \mathcal{BT} \times \text{Spec } \mathbb{Z}_p.$$

For each positive integer  $m$  we also consider the restriction

$$\pi^{(m)} : \text{Spec } A/p^m A \rightarrow \mathcal{BT} \times \text{Spec } \mathbb{Z}/p^m \mathbb{Z}.$$

We recall that the points of  $\overline{\mathcal{BT}}$  are pairs  $(k, H)$  where  $H$  is a  $p$ -divisible group over a field  $k$  of characteristic  $p$ , modulo the minimal equivalence relation such that  $(k, H) \sim (k', H')$  if there is  $k \rightarrow k'$  with  $H_{k'} \cong H'$ .

**Proposition 7.1.** *There is a pair  $(A, G)$  with the following properties:*

- (i) *The ring  $A$  is  $p$ -adic and excellent.*
- (ii) *For each maximal ideal  $\mathfrak{m}$  of  $A$ , the residue field  $A/\mathfrak{m}$  is perfect, and the group  $G \otimes \hat{A}_{\mathfrak{m}}$  is a universal deformation of its special fibre.*
- (iii) *All points of  $\overline{\mathcal{BT}}$  which correspond to isoclinic  $p$ -divisible groups lie in the image of  $\pi$ .*

**Proposition 7.2.** *Assume that  $(A, G)$  satisfies (i) and (ii). Then  $\pi$  is ind-smooth, and  $\pi^{(m)}$  is ind-smooth and quasi-étale. If (iii) holds as well, the morphisms  $\pi$  and  $\pi^{(m)}$  are also faithfully flat.*

We note the following consequence of Propositions 7.1 and 7.2.

**Corollary 7.3.** *There is a presentation  $\pi : \operatorname{Spec} A \rightarrow \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}_p$  such that  $A$  is an excellent  $p$ -adic ring,  $\pi$  ind-smooth, and the residue fields of the maximal ideals of  $A$  are perfect.*  $\square$

Let us prove Proposition 7.1. As explained in [NVW, Sec. 2], pairs  $(A, G)$  that satisfy (i)–(iii) can be constructed using integral models of suitable PEL-Shimura varieties. In that case,  $A/p^n A$  is smooth over  $\mathbb{Z}/p^n \mathbb{Z}$ . For completeness we give another construction using the Rapoport-Zink isogeny spaces of  $p$ -divisible groups.

*Proof of Proposition 7.1.* Let  $\mathbb{G}$  be a  $p$ -divisible group over  $\mathbb{F}_p$  which is descent in the sense of [RZ, Def. 2.13]. By [RZ, Thm. 2.16] there is a formal scheme  $M$  over  $\operatorname{Spf} \mathbb{Z}_p$  which is locally formally of finite type and which represents the following functor on the category of rings  $R$  in which  $p$  is nilpotent:  $M(R)$  is the set of isomorphism classes of pairs  $(G, \rho)$  where  $G$  is a  $p$ -divisible group over  $R$  and where  $\rho : \mathbb{G} \otimes R/pR \rightarrow G \otimes R/pR$  is a quasi-isogeny. Let  $G^{\operatorname{univ}}$  be the universal group over  $M$ .

If  $U = \operatorname{Spf} A$  is an affine open subscheme of  $M$ , then  $A$  is an  $I$ -adic noetherian  $\mathbb{Z}_p$ -algebra such that  $A/I$  is of finite type over  $\mathbb{F}_p$ . Thus  $A$  is  $p$ -adic and excellent; see [Va, Thm. 9]. The restriction of  $G^{\operatorname{univ}}$  to  $U$  defines a  $p$ -divisible group  $G$  over  $\operatorname{Spec} A$  because  $p$ -divisible groups over  $\operatorname{Spf} A$  and over  $\operatorname{Spec} A$  are equivalent; see [Me1, Lemma 4.16]. For each maximal ideal  $\mathfrak{m}$  of  $A$  the residue field  $A/\mathfrak{m}$  is finite, and the definition of  $M$  implies that  $G \otimes \hat{A}_{\mathfrak{m}}$  is a universal deformation. Thus (i) and (ii) hold.

For  $0 \leq d \leq h$  let  $\mathbb{G}_d$  be a decent isoclinic  $p$ -divisible group of height  $h$  and dimension  $d$  over  $\mathbb{F}_p$ , and let  $M_d$  be the associated isogeny space over  $\operatorname{Spf} \mathbb{Z}_p$ . It is well-known that there is a positive integer  $\delta$  depending on  $h$  such that if two  $p$ -divisible groups of height  $h$  over an algebraically closed field  $k$  are isogeneous, then there is an isogeny between them of degree at most  $\delta$ . Thus there is a finite set of affine open subschemes  $\operatorname{Spf} A_{d,i}$  of  $M_d$  such that each isoclinic  $p$ -divisible group of height  $h$  over  $k$  appears in the universal group over  $\operatorname{Spec} A_{d,i,\operatorname{red}}$  for some  $(d, i)$ . Let  $A$  be the product of all  $A_{d,i}$  and let  $G$  be the  $p$ -divisible group over  $A$  defined by the universal groups over  $A_{d,i}$ . Then  $(A, G)$  satisfies (i)–(iii).  $\square$

Let us prove the first part of Proposition 7.2.

**Lemma 7.4.** *If  $(A, G)$  satisfies (i) and (ii), then  $\pi$  is ind-smooth.*

*Proof.* Let  $X = \operatorname{Spec} A$  and  $\mathcal{X} = \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}_p$  and  $\mathcal{X}_n = \mathcal{BT}_n \times \operatorname{Spec} \mathbb{Z}_p$ . For  $T \rightarrow \mathcal{X}$  with affine  $T$ , the affine scheme  $X \times_{\mathcal{X}} T$  is the projective limit of the affine schemes  $X \times_{\mathcal{X}_n} T$ . Since ind-smooth homomorphisms are stable under direct limits of rings, to prove the lemma it suffices to show that the composition

$$\pi_n : X \xrightarrow{\pi} \mathcal{X} \rightarrow \mathcal{X}_n$$

is ind-smooth for each  $n \geq 1$ . By Popescu's theorem this means that  $\pi_n$  is regular. By EGA IV, 6.8.3 this holds if and only if for each closed point  $x \in X$  the composition

$$\hat{X} = \operatorname{Spec} \hat{\mathcal{O}}_{X,x} \rightarrow X \rightarrow \mathcal{X}_n$$

is regular. Let  $k$  be the finite residue field of  $x$ . Let  $Y_0 \rightarrow \mathcal{X}_n$  be a smooth presentation such that  $\pi_n(x)$  lifts to a  $k$ -valued point  $y_0$  of  $Y_0$ . Let  $Y = Y_0 \otimes_{\mathbb{Z}_p} W(k)$  and choose a closed point  $y \in Y$  over  $y_0$  with residue field  $k$ . Let  $\hat{Y} = \operatorname{Spec} \hat{\mathcal{O}}_{Y,y}$ . Since  $Y \rightarrow \mathcal{X}_n$  is smooth,  $\hat{Y} \rightarrow \mathcal{X}_n$  is regular. There is a  $p$ -divisible group  $H$  over  $\hat{Y}$  such that the special fibre  $H_y$  is equal to  $G_x$  and such that the truncation  $H[p^n]$  is the inverse image of the universal group over  $\mathcal{X}_n$ ; see [Il2, Thm. 4.4]. There is a unique morphism  $\psi : \hat{Y} \rightarrow \hat{X}$  such that  $\psi^* G_{\hat{X}}$  is equal to  $H$  as deformations of  $G_x$  because  $G_{\hat{X}}$  is assumed to be universal.

$$\begin{array}{ccccccc} \hat{X} & \longrightarrow & X & \xrightarrow{G} & \mathcal{X} & \longrightarrow & \mathcal{X}_n \\ & \swarrow \psi & & \nearrow H & & \uparrow & \\ & & \hat{Y} & \longrightarrow & Y & & \end{array}$$

Since  $\mathcal{X}_n$  is smooth over  $\mathbb{Z}_p$  by [Il2, Thm. 4.4],  $X$  and  $Y_0$  are smooth over  $\mathbb{Z}_p$ . Thus  $\hat{\mathcal{O}}_{X,x}$  and  $\hat{\mathcal{O}}_{Y,y}$  are power series rings over  $W(k)$ . The diagram induces a diagram of the tangent spaces at the images of the closed point of  $\hat{Y}$ . Here  $\hat{X} \rightarrow \mathcal{X}$  is bijective on the tangent spaces since  $G_{\hat{X}}$  is a universal deformation,  $\mathcal{X} \rightarrow \mathcal{X}_n$  is bijective on the tangent spaces by [Il2, Thm. 4.4], and  $\hat{Y} \rightarrow \mathcal{X}_n$  is surjective on the tangent spaces since  $Y \rightarrow \mathcal{X}_n$  is smooth. Thus  $\psi$  is surjective on the tangent spaces, which implies that  $\hat{\mathcal{O}}_{Y,y}$  is a power series ring over  $\hat{\mathcal{O}}_{X,x}$ ; in particular  $\psi$  is faithfully flat. Since  $\hat{Y} \rightarrow \mathcal{X}_n$  is regular it follows that  $\hat{X} \rightarrow \mathcal{X}_n$  is regular.  $\square$

Now we prove the second part of Proposition 7.2. This is not used later.

**Lemma 7.5.** *If  $(A, G)$  satisfies (i) and (ii), then  $\pi^{(m)}$  is quasi-étale.*

*Proof.* Let  $X^{(m)} = \operatorname{Spec} A/p^m A$  and  $\mathcal{X}^{(m)} = \mathcal{BT} \times \operatorname{Spec} \mathbb{Z}/p^m \mathbb{Z}$ . For a morphism  $Y \rightarrow \mathcal{X}^{(m)}$  with affine  $Y$  let  $Z = X \times_{\mathcal{X}} Y$  and let  $\pi' : Z \rightarrow Y$  be the second projection. We have to show that  $L_{Z/Y}$  is acyclic. Here  $\pi'$  is ind-smooth by Lemma 7.4. Thus  $L_{Z/Y}$  is isomorphic to  $\Omega_{Z/Y}$ , and it suffices to show that  $\pi^{(m)}$  is formally unramified.

For an arbitrary morphism  $g : \text{Spec } B \rightarrow \mathcal{X}^{(m)}$ , given by a  $p$ -divisible group  $H$  over  $B$ , we write  $\Lambda_H = \text{Lie}(H) \otimes_B \text{Lie}(H^\vee)$ . There is a Kodaira-Spencer homomorphism  $\kappa_H : \Lambda_H \rightarrow \Omega_B$ , which is surjective if and only if  $g$  is formally unramified. For a closed point  $x \in X^{(m)}$  let  $A_x$  be the complete local ring at  $x$  and let  $\hat{X}^{(m)} = \text{Spec } A_x$ . We have  $i : \hat{X}^{(m)} \rightarrow X^{(m)}$ . There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & i^* \Omega_{X^{(m)}} & \longrightarrow & \Omega_{\hat{X}^{(m)}} & \longrightarrow & \Omega_{\hat{X}^{(m)}/X^{(m)}} \longrightarrow 0 \\ & & \downarrow i^* \kappa_G & & \downarrow \kappa_{i^* G} & & \\ & & i^* \Lambda_G & \xlongequal{\quad} & \Lambda_{i^* G} & & \end{array}$$

The upper line is exact because  $\hat{X}^{(m)} \rightarrow X^{(m)}$  is regular, thus ind-smooth, which implies that  $\Omega_{\hat{X}^{(m)}/X^{(m)}}$  is concentrated in degree zero. Since  $A_x$  is isomorphic to a power series ring  $W_m(k)[[t_1, \dots, t_r]]$  for a perfect field  $k$ , the  $A_x$ -module  $\Omega_{\hat{X}^{(m)}}$  is free with basis  $dt_1, \dots, dt_r$ . These elements appear in  $i^* \Omega_{X^{(m)}}$ , and thus  $\Omega_{\hat{X}^{(m)}/X^{(m)}}$  is zero. The homomorphism  $\kappa_{i^* G}$  is an isomorphism because  $i^* G$  is assumed to be universal. Thus  $i^* \kappa_G$  is an isomorphism for all  $x$ , which implies that  $\kappa_G$  is an isomorphism, and  $\pi^{(m)}$  is formally unramified as desired.  $\square$

Finally we prove the last part of Proposition 7.2.

**Lemma 7.6.** *Assume that  $(A, G)$  satisfies (i) and (ii). Then  $\pi$  is surjective if and only if (iii) holds.*

*Proof.* It suffices to prove the implication  $\Leftarrow$ . We write  $X = \text{Spec } A$  and  $\mathcal{X} = \mathcal{BT} \times \text{Spec } \mathbb{Z}_p$ . Let  $k$  be an algebraically closed field of characteristic  $p$  and let  $H : \text{Spec } k \rightarrow \mathcal{X}$  be a geometric point, i.e. a  $p$ -divisible group  $H$  over  $k$ . It suffices to show that  $X_H = X \times_{\mathcal{X}} \text{Spec } k$  is non-empty. Let  $K$  be an algebraic closure of  $k((t))$  and let  $R$  be the ring of integers in  $K$ . As in the proof of Lemma 4.6 we find a  $p$ -divisible group  $H'$  over  $R$  with generic fibre  $H \otimes_k K$  and with isoclinic special fibre. We consider the morphism  $\text{Spec } R \rightarrow \mathcal{X}$  defined by  $H'$ . The projection  $X \times_{\mathcal{X}} \text{Spec } R \rightarrow \text{Spec } R$  is flat because it is ind-smooth by Lemma 7.4, and its image contains the closed point of  $\text{Spec } R$  by (iii). Thus the projection is surjective, which implies that  $X \times_{\mathcal{X}} \text{Spec } K = X_H \otimes_k K$  is non-empty; thus  $X_H$  is non-empty.  $\square$

*Proof of Proposition 7.2.* Use Lemmas 7.4, 7.5, and 7.6.  $\square$

## 8. RELATION WITH THE FUNCTOR BT

For a  $p$ -adic ring  $R$  we consider the following commutative diagram of categories, where f.=formal, g.=groups, n.=nilpotent. The vertical arrows are the inclusions. The functor  $\text{BT}_R$  is defined in [Zi1, Thm. 81]. Its restriction



to nilpotent displays gives formal  $p$ -divisible groups by [Zi1, Cor. 89].

$$\begin{array}{ccccccc}
 (p\text{-div}/R) & \xrightarrow{\Phi_R} & (\text{disp}/R) & \xrightarrow{\text{BT}_R} & (\text{f. g.}/R) \\
 \uparrow & & \uparrow & & \uparrow \\
 (\text{f. } p\text{-div}/R) & \xrightarrow{\Phi_R^1} & (\text{n. disp}/R) & \xrightarrow{\text{BT}_R^1} & (\text{f. } p\text{-div}/R) & \xrightarrow{\Phi_R^1} & (\text{n. disp}/R)
 \end{array}$$

Here  $\text{BT}_R^1$  is an equivalence by [Zi1] if  $R$  is excellent and by [La1] in general, and  $\Phi_R^1$  is an equivalence by Theorem 5.1.

**Lemma 8.1.** *There is a natural isomorphism of functors  $\Phi_R^1 \circ \text{BT}_R^1 \cong \text{id}$ .*

We have a functor  $\Upsilon_R : (\text{disp}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathscr{W}_R)$ .

Let  $\Upsilon_R^1 : (\text{n. disp}/R) \rightarrow (\text{filtered } F\text{-}V\text{-modules over } \mathscr{W}_R)$  be its restriction.

*Proof of Lemma 8.1.* By [Zi1, Thm. 94 and Cor. 97], for each nilpotent display  $\mathscr{P}$  over  $R$  there is an isomorphism, functorial in  $\mathscr{P}$  and in  $R$ ,

$$u_R(\mathscr{P}) : \Upsilon_R^1(\Phi_R^1(\text{BT}_R^1(\mathscr{P}))) \cong \Upsilon_R^1(\mathscr{P}).$$

We have to show that  $u_R(\mathscr{P})$  commutes with  $F_1$ . This is automatic if  $R$  has no  $p$ -torsion because then  $W(R)$  has no  $p$ -torsion, and  $pF_1 = F$ . Since  $\Phi_R^1$  is an equivalence, we may assume that  $\mathscr{P} = \Phi_R^1(G)$  for a formal  $p$ -divisible group  $G$  over  $R$ . We may also assume that  $p$  is nilpotent in  $R$ .

Let  $\text{Spec } A \rightarrow \mathscr{BT} \times \text{Spec } \mathbb{Z}_p$  be a presentation given by a  $p$ -divisible group  $H$  over  $A$  such that  $A$  is noetherian; see Corollary 7.3. Let  $J \subset A$  be the ideal such that  $X \times_{\mathscr{BT}} \mathscr{BT}^o = \text{Spec } A/J$  and let  $\hat{A}$  be the  $J$ -adic completion of  $A$ . If  $A$  is constructed using isogeny spaces as in the proof of Proposition 7.1, then  $A$  is  $I$ -adic for an ideal  $I$  which contains  $J$ , and thus  $A$  is already  $J$ -adic. In any case, since  $\mathbb{Z}_p \rightarrow A \rightarrow \hat{A}$  is flat,  $\hat{A}$  has no  $p$ -torsion. Thus the compatible system  $[u_{A/J^n}(\Phi_{A/J^n}^1(H))]_{n \geq 1}$  necessarily preserves  $F_1$ .

For  $G$  over  $R$  as above we consider  $\text{Spec } R \times_{\mathscr{BT}} \text{Spec } A = \text{Spec } S$ . Since  $\text{Spec } R_{\text{red}} \rightarrow \text{Spec } R \rightarrow \mathscr{BT}$  factors over  $\mathscr{BT}^o$ , the ideal  $JS$  is a nilideal. Thus  $JS$  is nilpotent as  $J$  is finitely generated by Lemma 1.9; hence for sufficiently large  $n$  we have  $\text{Spec } S = \text{Spec } R \times_{\mathscr{BT}} \text{Spec } A/J^n$ . By construction,  $R \rightarrow S$  is faithfully flat, thus injective, and  $G \otimes_R S \cong H \otimes_A S$ . Thus  $u_R(\Phi_R^1(G))$  preserves  $F_1$  since this holds for  $u_{A/J^n}(\Phi_{A/J^n}^1(H))$ .  $\square$

*Remark 8.2.* The above proof of Lemma 8.1 uses that  $\Phi_R^1$  is an equivalence, but this could be avoided by an more careful (elementary) analysis of the stack of displays over rings in which  $p$  is nilpotent. Thus the facts that  $\Phi_R^1$  and  $\text{BT}_R^1$  are equivalences can be derived from each other.

Since  $\Phi_R^1$  is an equivalence, the isomorphism  $\Phi_R^1 \circ \text{BT}_R^1 \cong \text{id}$  of Lemma 8.1 induces for each formal  $p$ -divisible group  $G$  over a  $R$  an isomorphism

$$\rho_R(G) : G \cong \text{BT}_R^1(\Phi_R^1(G)).$$

**Theorem 8.3.** *For each  $p$ -divisible group  $G$  over a  $p$ -adic ring  $R$  there is a unique isomorphism which is functorial in  $G$  and  $R$*

$$\tilde{\rho}_R(G) : \hat{G} \cong \text{BT}_R(\Phi_R(G))$$

*which coincides with  $\rho_R(G)$  if  $G$  is infinitesimal.*

If  $G$  is an extension of an étale  $p$ -divisible group by an infinitesimal  $p$ -divisible group, Theorem 8.3 follows from Lemma 8.4 below because both sides of  $\tilde{\rho}_R(G)$  preserve short exact sequences.

**Lemma 8.4.** *If  $G$  is étale, then  $\mathrm{BT}_R(\Phi_R(G))$  is zero.*

*Proof.* Let  $\Phi_R(G) = \mathcal{P} = (P, Q, F, F_1)$ . We have  $P = Q$ , and  $F_1 : P \rightarrow P$  is an  $f$ -linear isomorphism. Let  $N$  be a nilpotent  $R/p^n R$ -algebra for some  $n$ . By the definition of  $\mathrm{BT}_R$ , the group  $\mathrm{BT}_R(\mathcal{P})(N)$  is the cokernel of the endomorphism  $F_1 - 1$  of  $\hat{W}(N) \otimes_{W(R)} P$ . This endomorphism is bijective because here  $F_1$  is nilpotent since  $f$  is nilpotent on  $\hat{W}(N)$ .  $\square$

*Proof of Theorem 8.3.* Assume that  $G$  is a  $p$ -divisible group over an  $I$ -adic ring  $A$  such that  $G_{A/I}$  is infinitesimal. Then  $G_n = G_{A/I^n}$  is infinitesimal as well. Since formal Lie groups over  $A$  are equivalent to compatible systems of formal Lie groups over  $A/I^n$  for  $n \geq 1$ , the isomorphisms  $\rho_{A/I^n}(G_n)$  define the desired isomorphism  $\tilde{\rho}_A(G)$ , which is clearly unique. The construction is functorial in the triple  $(A, I, G)$ .

Assume that in addition a ring homomorphism  $u : A \rightarrow B$  is given such that  $B$  is  $J$ -adic and such that  $G_{B/J}$  is ordinary; we do not assume that  $u(I) \subseteq J$ . There is a unique exact sequence of  $p$ -divisible groups over  $B$

$$0 \rightarrow H \xrightarrow{\alpha} G_B \rightarrow H' \rightarrow 0$$

such that  $H$  is of multiplicative type and  $H'$  is étale. Consider the following diagram of isomorphisms; cf. Lemma 8.4.

$$(8.1) \quad \begin{array}{ccc} \hat{H} & \xrightarrow{\tilde{\rho}_B(H)} & \mathrm{BT}_B(\Phi_B(H)) \\ \alpha \downarrow & & \downarrow \alpha \\ \hat{G}_B & \xrightarrow{(\tilde{\rho}_A(G))_B} & (\mathrm{BT}_A(\Phi_A(G)))_B \end{array}$$

Since the construction of  $\tilde{\rho}_A(G)$  is functorial in  $(A, I)$ , the diagram commutes if  $u(I) \subseteq J$ . If Theorem 8.3 holds, (8.1) commutes always. We show directly that (8.1) commutes in a special case that allows to define  $\tilde{\rho}_R$  in general by descent. As in the proof of Proposition 7.1 we consider a decent  $p$ -divisible group  $\mathbb{G}$  over a perfect field  $k$  and an affine open subscheme  $U = \mathrm{Spf} A$  of the isogeny space of  $\mathbb{G}$  over  $\mathrm{Spf} W(k)$ . Let  $U_{\mathrm{red}} = \mathrm{Spec} A/I$  and let  $G$  be the universal  $p$ -divisible group over  $A$ . By passing to a connected component of  $U$  we may assume that  $A$  is integral. Since  $A$  has no  $p$ -torsion and since  $A/pA$  is regular,  $pA$  is a prime ideal. Let  $B = \hat{A}_{pA}$  be the complete local ring of  $A$  at this prime and let  $J = pB$ .

**Lemma 8.5.** *For this choice of  $(A, I, B, J, G)$  the diagram (8.1) is defined and commutes.*

*Proof.* In order that (8.1) is defined we need that  $G_{B/J}$  is ordinary. Then the defect of commutativity of (8.1) is an automorphism  $\xi = \xi(A, I, B, J, G)$  of  $\hat{H}$ , which is functorial with respect to  $(A, I, B, J, G)$ .

Step 1. For an arbitrary maximal ideal  $\mathfrak{m}$  of  $A$  let  $A_1 = \hat{A}_{\mathfrak{m}}$ , let  $I_1$  be the maximal ideal of  $A_1$ , let  $B_1$  be the complete local ring of  $A_1$  at the

prime ideal  $pA_1$ , and let  $J_1 = pB_1$ . We have compatible injective homomorphisms  $A \rightarrow A_1$  with  $I \rightarrow I_1$  and  $B \rightarrow B_1$  with  $J \rightarrow J_1$ . Moreover  $A_1 \cong W(k_1)[[t_1, \dots, t_r]]$  for a finite extension  $k_1$  of  $k$ , and  $G_{A_1}$  is a universal deformation. Thus  $G_{B_1/J_1}$  is ordinary, which implies that  $G_{B/J}$  is ordinary, and it suffices to show that  $\xi(A_1, I_1, B_1, J_1, G_{A_1}) = \text{id}$ .

Step 2. Next we achieve  $r = 1$  by a blowing-up construction. Let  $k_2$  be an algebraic closure of the function field  $k_1(\{t_i/t_j\}_{1 \leq i, j \leq r})$ , let  $A_2 = W(k_2)[[t]]$ , let  $I_2 = tA_2$ , let  $B_2$  be the completion of  $A_2$  at the prime ideal  $pA_2$ , and let  $J_2 = pB_2$ . Thus  $B_2/J_2 = k_2((t))$ . There is a natural local homomorphism  $A_1 \rightarrow A_2$  with  $t_i \mapsto [t_i/t_1]t$ , which induces an injective homomorphism  $B_1 \rightarrow B_2$ . Thus it suffices to show that  $\xi(A_2, I_2, B_2, J_2, G_{A_2}) = \text{id}$ .

Step 3. Let  $L$  be the completion of an algebraic closure of  $k_2((t))$  and let  $\mathcal{O} \subset L$  be its ring of integers. For a fixed  $n \geq 1$  let  $A' = W_n(\mathcal{O})$  and let  $I'$  be the kernel of  $A' \rightarrow \mathcal{O}/t\mathcal{O}$ , thus  $I'$  is generated by  $([t], p)$ . It is easy to see that  $A'$  is  $I'$ -adic, using that  $\mathcal{O}$  is  $t$ -adic and that  $p^r[t] = v^r[t^{p^r}]$ . Let  $B' = W_n(L)$  and  $J' = pB'$ . The homomorphism  $A_2 \rightarrow A'$  defined by the inclusion  $W(k_2) \rightarrow W(L)$  and by  $t \mapsto [t]$  induces a homomorphism  $B_2 \rightarrow B'$  such that the kernels of  $B_2 \rightarrow B'$  for increasing  $n$  have zero intersection. Thus it suffices to show that  $\xi(A', I', B', J', G_{A'}) = \text{id}$ .

Step 4. Since  $L$  is algebraically closed,  $H_L$  is isomorphic to  $\mu_{p^\infty}^d$ . Since  $G_L$  is ordinary, the inclusion  $H_L \rightarrow G_L$  splits uniquely; i.e. we have a homomorphism  $\psi_L : G_L \rightarrow \mu_{p^\infty}^d$  that induces an isomorphism of the formal completions. Since  $\mathcal{O}$  is normal, the Serre dual of  $\psi_L$  extends to a homomorphism over  $\mathcal{O}$ , thus  $\psi_L$  extends to a homomorphism  $\psi_{\mathcal{O}} : G_{\mathcal{O}} \rightarrow \mu_{p^\infty}^d$ . The homomorphism  $p^n \psi_{\mathcal{O}}$  extends to  $\psi' : G \rightarrow \mu_{p^\infty}^d$  over  $A'$ , and its restriction  $\psi'_B : G_{B'} \rightarrow \mu_{p^\infty}^d$  over  $B$  induces an isogeny of the multiplicative parts, which commutes with the associated  $\xi$ 's by functoriality. Thus it suffices to show that  $\xi' = \xi(A', I', B', J', \mu_{p^\infty}) = \text{id}$ .

Since  $A'$  is  $p$ -adic and since  $\mu_{p^\infty}$  is infinitesimal over  $A'/pA'$ , the element  $\xi'' = \xi(A', pA', B', J', \mu_{p^\infty})$  is well-defined, and it induces  $\xi'$  by functoriality. But we have  $\xi'' = \text{id}$  because  $A' \rightarrow B'$  maps  $pA'$  into  $J' = pB'$ . This proves Lemma 8.5.  $\square$

We continue the proof of Theorem 8.3 and write  $\text{BT}_R(\Phi_R(G)) = G^+$ . Let  $\text{Spec } A \rightarrow \mathcal{BT} \times \text{Spec } \mathbb{Z}_p$  be the presentation constructed in the proof of Proposition 7.1 and let  $G$  be the universal group over  $A$ . The ring  $A$  is  $I$ -adic such that  $G_{A/I}$  is isoclinic. Thus the above construction applies and gives  $\tilde{\rho}_A(G) : \hat{G} \cong G^+$ ; here the components of  $\text{Spec } A$  where  $G$  is étale do not matter in view of Lemma 8.4. Let  $\text{Spec } A \times_{\mathcal{BT} \times \text{Spec } \mathbb{Z}_p} \text{Spec } A = \text{Spec } C$  and let  $\hat{C}$  be the  $p$ -adic completion of  $C$ .

For an arbitrary ring  $R$  in which  $p$  is nilpotent we want to define  $\tilde{\rho}_R$  by descent, starting from  $\tilde{\rho}_A(G)$ . This is possible if and only if the inverse images of  $\tilde{\rho}_A(G)$  under the two projections  $p_i : \text{Spec } \hat{C} \rightarrow \text{Spec } A$  coincide, i.e. if the following diagram of formal Lie groups over  $\hat{C}$  commutes, where

$u : p_1^*G \cong p_2^*G$  is the given descent isomorphism.

$$(8.2) \quad \begin{array}{ccc} p_1^*\hat{G} & \xrightarrow{\hat{u}} & p_2^*\hat{G} \\ p_1^*(\bar{\rho}) \downarrow & & \downarrow p_2^*(\bar{\rho}) \\ p_1^*G^+ & \xrightarrow{u^+} & p_2^*G^+ \end{array}$$

Let  $A = \prod A_i$  be a maximal decomposition so that each  $A_i$  is a domain, let  $B_i$  be the complete local ring of  $A_i$  at the prime  $pA_i$ , and let  $B = \prod B_i$ . Since the two homomorphisms  $A \rightarrow C$  are flat and since  $A \rightarrow B$  is flat and induces an injective map  $A/p^nA \rightarrow B/p^nB$ , the natural homomorphism  $C \rightarrow B \otimes_A C \otimes_A B = C'$  induces an injective homomorphism of the  $p$ -adic completions  $\hat{C} \rightarrow \hat{C}'$ . Thus the commutativity of (8.2) can be verified over  $\hat{C}'$ . Let  $H$  be the multiplicative part of the ordinary  $p$ -divisible group  $G_B$ .

Since the construction of  $\tilde{\rho}$  is functorial with respect to the projections of  $p$ -adic rings  $q_1, q_2 : \text{Spec } \hat{C}' \rightarrow \text{Spec } B$ , the following diagram of formal Lie groups over  $\hat{C}'$  commutes.

$$(8.3) \quad \begin{array}{ccc} q_1^*\hat{H} & \xrightarrow{\hat{u}} & q_2^*\hat{H} \\ q_1^*(\bar{\rho}) \downarrow & & \downarrow q_2^*(\bar{\rho}) \\ q_1^*H^+ & \xrightarrow{u^+} & q_2^*H^+ \end{array}$$

Lemma 8.5 implies that the inclusion  $H \rightarrow G_B$  induces an isomorphism of diagrams  $(8.3) \cong (8.2) \otimes_{\hat{C}} \hat{C}'$ . Thus (8.2) commutes as well.  $\square$

**8.1. Complement to [La1].** The proof that  $\text{BT}_R^1$  is an equivalence in [La1] proceeds along the following lines. First, by [Zi1] the functor is always faithful, and fully faithful if  $R$  is reduced over  $\mathbb{F}_p$ . Second, by using an  $\infty$ -smooth presentation of  $\mathcal{BT}^o$  as in Lemma 1.6, one deduces that  $\text{BT}_R^1$  is essentially surjective if  $R$  is reduced over  $\mathbb{F}_p$ . Using this, one shows that  $\text{BT}_R^1$  is fully faithful in general, and the general equivalence follows.

The second step is based on the following consequence of the first step. A faithfully flat homomorphism of reduced rings  $R \rightarrow S$  is called an *admissible covering* if  $S \otimes_R S$  is reduced. In this case, a formal  $p$ -divisible group  $G$  over  $R$  lies in the image of  $\text{BT}_R^1$  if  $G_S$  lies in the image of  $\text{BT}_S^1$ . In [La1, Sec. 3], some effort is needed to find a sufficient supply of admissible coverings.

The proof can be simplified as follows if one starts with a reduced presentation  $\pi : \text{Spec } A' \rightarrow \mathcal{BT} \times \text{Spec } \mathbb{Z}_p$  with excellent  $A'$  such that  $A'/\mathfrak{m}$  is perfect for all maximal ideals  $\mathfrak{m}$  of  $A'$ ; see Corollary 7.3. Let  $\text{Spec } A \rightarrow \mathcal{BT}^o$  be the restriction of  $\pi$ . As in [La1] it suffices to show that the universal group  $G$  over  $A$  lies in the image of  $\text{BT}_A^1$ . Since  $A$  is excellent, this follows from [Zi1]. A direct argument goes as follows: The homomorphisms  $A \rightarrow \prod_{\mathfrak{m}} A_{\mathfrak{m}}$  and  $A_{\mathfrak{m}} \rightarrow \hat{A}_{\mathfrak{m}}$  are admissible coverings, which reduces the surjectivity of  $\text{BT}_A^1$  to the surjectivity of  $\text{BT}_{A/\mathfrak{m}^n}^1$ . By deformation theory this is reduced to the case of the perfect fields  $A/\mathfrak{m}$ , which is classical.

**8.2. Erratum to [La1].** [La1, Lemma 3.3] asserts that if  $R$  is a noetherian ring and if  $R \rightarrow S$  and  $S \rightarrow T$  are admissible coverings, then  $R \rightarrow T$  is an admissible covering too. This is false; see Example 8.7 below. The proof assumes incorrectly that a field extension  $L/K$  such that  $L \otimes_K L$  is reduced must be separable. The following part loc. cit. is proved correctly.

**Lemma 8.6.** *Let  $R \rightarrow S$  be a faithfully flat homomorphism of reduced rings where  $R$  is noetherian such that for all minimal prime ideals  $\xi \subset S$  and  $\eta = \xi \cap R$  the field extension  $R_\eta \rightarrow S_\xi$  is separable. Then  $S \otimes_R S$  is reduced.*

The incorrect [La1, Lemma 3.3] is only used in the proof of [La1, Prop. 3.4], where it can be avoided as follows. For certain rings  $A \rightarrow \hat{A} \rightarrow \hat{B}$  one needs that  $\hat{B} \otimes_{\hat{A}} \hat{B}$  is reduced. The proof shows that  $A \rightarrow \hat{A}$  and  $\hat{A} \rightarrow \hat{B}$  satisfy the hypotheses of Lemma 8.6. Thus  $A \rightarrow \hat{B}$  satisfies these hypotheses as well, and the assertion follows.

**Example 8.7.** Let  $K$  be a field of characteristic  $p$  and let  $a, b, c$  be part of a  $p$ -basis of  $K^{1/p}$  over  $K$ . Let  $L = K(X, a + bX)$  and  $M = L(Y, a + cY)$  where  $X$  and  $Y$  are algebraically independent over  $K$ . Then  $L \otimes_K L$  and  $M \otimes_L M$  are reduced, but  $M \otimes_K M$  is not reduced. In particular,  $L$  is not separable over  $K$ .

## REFERENCES

- [Be] P. Berthelot: Théorie de Dieudonné sur un anneau de valuation parfait. Ann. Sci. Ec. Norm. Sup. (4) **13** (1980), 225–268
- [BBM] P. Berthelot, L. Breen, and W. Messing: Théorie de Dieudonné cristalline II. Lecture Notes in Math. **930**, Springer Verlag, 1982
- [BM] P. Berthelot, W. Messing: Théorie de Dieudonné cristalline, III. The Grothendieck Festschrift, Vol. I, 173–247, Progr. Math. **86**, Birkhäuser, 1990
- [G1] A. Grothendieck: Groupes de Barsotti-Tate et cristaux. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 431–436. Gauthier-Villars, Paris, 1971
- [G2] A. Grothendieck: Groupes de Barsotti-Tate et Cristaux de Dieudonné. Université de Montréal, 1974
- [II1] L. Illusie: Complexe cotangent et déformations, I. Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin-New York, 1971
- [II2] L. Illusie: Déformations de groupes de Barsotti-Tate (d’après A. Grothendieck). In *Seminar on arithmetic bundles: the Mordell conjecture*, Astérisque **127** (1985), 151–198
- [Ka] N. M. Katz: Slope filtration of  $F$ -crystals. *Journées de Géométrie Algébrique de Rennes*, Astérisque **63** (1979), 113–163
- [Ki] M. Kisin: Crystalline representations and  $F$ -crystals, *Algebraic geometry and number theory*, 459–496, Progr. Math., Vol. 253, Birkhäuser, 2006
- [LZ] A. Langer and Th. Zink: De Rham-Witt cohomology for a proper and smooth morphism. J. Inst. Math. Jussieu **3** (2004), no. 2, 231–314
- [La1] E. Lau: Displays and formal  $p$ -divisible groups. Invent. Math. **171** (2008), 617–628
- [La2] E. Lau: Frames and finite group schemes over complete regular local rings. arXiv:0908.4588, to appear in Doc. Math.
- [La3] E. Lau: A relation between crystalline Dieudonné theory and Dieudonné displays, 2010, available at <http://arxiv.org/>
- [Ma] H. Matsumura: *Commutative ring theory*. Cambridge Univ. Press, 1986
- [Me1] W. Messing: The crystals associated to Barsotti-Tate groups: with applications to abelian schemes. Lecture Notes in Math. **264**, Springer Verlag, 1972
- [Me2] W. Messing: Travaux de Zink. Séminaire Bourbaki 2005/2006, exp. 964, Astérisque **311** (2007), 341–364

- [NVW] M.-H. Nicole, A. Vasiu, and T. Wedhorn, Purity of level  $m$  stratifications. [arxiv.org:0808.1629](https://arxiv.org/abs/0808.1629), to appear in Ann. Sci. Ec. Norm. Sup.
- [O1] F. Oort: Newton polygons and formal groups: conjectures by Manin and Grothendieck. Ann. of Math. **152** (2000), 183–206
- [Po] D. Popescu: General Néron desingularization and approximation. Nagoya Math. J. **104** (1986), 85–115
- [RZ] M. Rapoport, Th. Zink: Period spaces of  $p$ -divisible groups. Ann. Math. Stud. **141**, Princeton Univ. Press, 1996
- [Sw] R. Swan: Néron-Popescu desingularization. In: Algebra and geometry (Taipei, 1995), 135–192, Lect. Algebra Geom. **2**, Int. Press, Cambridge, MA, 1998
- [Va] P. Valabrega: A few theorems on completion of excellent rings. Nagoya Math. J. **61** (1976), 127–133
- [W] T. Wedhorn: The dimension of Oort strata of Shimura varieties of PEL-type. In: Moduli of Abelian Varieties, Progr. Math. Vol. 195, 441–471, Birkhäuser, Basel, 2001
- [Zi1] Th. Zink: The display of a formal  $p$ -divisible group. In: Cohomologies  $p$ -adiques et applications arithmétiques, I, Astérisque **278** (2002), 127–248

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